1X QL

ANALYSIS
Per Quantitatum
SERIES, FLUXIONES,
AC
DIFFERENTIAS:
CUM
Enumeratione Linearum
TERTII ORDINIS.

92 l

MUSEUM BRITANNICUM

k

I. Newton (Sir I)

LONDINI:
Ex Officina Pearsoniana. *Anno* M.DCC.XI.

12

MVSEVM
BRITAN
NICVM

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Præfatio Editoris.

TRactatus aliquot Mathematicos Viri Illuſtriſſimi D. Newtoni *in lucem edimus, quorum primus & ultimus nunc primum prodeunt, reliqui vero vel à ſe vel ab aliis ante hac publici juris facti ſunt. Hæc autem Editio caſui, ſed tamen non ſine ipſius conſenſu prius impetrato, ortum acceptum refert.*

Etenim ſecundus jam agitur annus ex quo ſcrinia D. Collinſii (*qui, uti notum eſt, ampliſſimum cum ſui ſæculi Mathematicis commercium habuit*) *meas in manus inciderunt; & in illis plurima reperi à cunctis fere totius* Europæ *eruditis ipſi communicata; & inter ea non pauca, quæ à Viro Cl* D. Newtono *ſcripta fuerunt; quæ cum tantæ molis eſſent, ut ſimul Tractatum breviuſculum poſſent conficere, cæpi de iis* *edendis cogitare. Quum autem animadvertiſſem ſcripta ejus quæ jam in lucem prodierunt ferme idem cum hiſce argumentum habere, haud operæ pretium me facturum, ſi typis mandarem, exiſtimavi.*

Unus tamen erat brevis de Curvarum Quadratura Tractatus adeo luculenter & concinne ſcriptus, atque ita accommodatus ad inſtituendos eos, qui nondum totam iſtam Methodum perſpectam habeant, ut abſtinere non potuerim, quominus Auctoris licentiam eundem edendi peterem. Quam Ille non ſolum ſumma cum humanitate conceſſit; ſed & inſuper veniam dedit reliqua ipſius colligendi, quæ ad idem argumentum ſpectabant.

Hicce Tractatus, quem D. Collinſii *manu exaratum comperi, inſcriptus fuit* De Analyſi per Æquationes Infinitas, *& licet neque Auctoris nomen, neque tempus quo ſcriptus fuerat ullibi comparuerit; multa tamen continere ad D.* Newtoni *Methodos ſpectantia ſtatim agnovi, utpote Epiſtolarum autographo illi ad amuſſim reſpondentia, quod antea ipſe D.* Newtonus *ad D.* Oldenburgum *miſerat. Unde ſuſpicabar totum ex eodem fonte emanaſſe; nihilominus ſuſpenſo eram animo, uſque dum inventis, inter eaſdem chartas, Epiſtolis aliquot D.* Barrovii *ad D.* Collinſium *ſcriptis; tres reperi ad hunc Tractatum*

ctatum immediate ſpectantes; ex quibus facile intellexi quomodo D. Collinſius *eum obtinuerat.*

In una Epiſtolarum e Collegio S. S. Tr. data 20 Julii 1669, *verſus finem* D. Barrovius *hæc ſcribit.*

* Amicus quidam apud nos commorans, qui eximio in his rebus pollet ingenio, nudiuſtertius chartas quaſdam mihi tradidit, in quibus Magnitudinum dimenſiones ſupputandi Methodos, *Mercatoris* methodo ſimiles, maxime vero Generales, deſcripſit; ſimulque Æquationes reſolvendi, quæ, ut opinor, tibi placebunt, quas una cum proximis literis ad te mittam.

In hac Epiſtola narrat argumentum chartarum Amici ſui fuiſſe Computationem dimenſionum Magnitudinum, & Æquationum reſolutionem, quod cum Tractatus hujus argumento quadrat.

Verſus finem alterius Epiſtolæ ad D. Collinſium *e Collegio S. S. Tr. datæ ult.* Julii 1669, D. Barrovius *ita ſcribit.*

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* A Friend of mine here, that hath an excellent Genius to theſe things, brought me the other Day, ſome Papers, wherein he hath ſet down Methods of Calculating the Dimenſions of Magnitudes, like that of Mr. *Mercator*, but very general, as alſo of Reſolving Equations, which I ſuppoſe will pleaſe you, and I ſhall ſend you them by the next.

> [†] Mitto quas pollicitus eram Amici chartas, quæ uti ſpero haud parum te oblectabunt. Remittas, quæſo, quum eas quantum tibi viſum fuerit perlegeris; id enim poſtulavit Amicus meus, cum primum eum rogavi, ut eas tecum communicare mihi liceret. Quantocyus igitur, obſecro, te eas recepiſſe fac me certiorem, quod illis metuo, quippe qui eas per Veredarium publicum ad te mittere non dubitaverim, quo tibi morem gererem quam citiſſime.

Ex hac Epiſtola conſtat D. Barrovium *dictum Librum miſiſſe, ea lege, ut ſibi remitteretur. Unde manifeſtum eſt, quare Tractatus, quem inveni,* D. Collinſii *manu ſcriptus fuit, autographo nempe ipſi Auctori reſtituto.*

In tertia a D. Barrovio *ad* D. Collinſium *Epiſtola data* 20 Auguſti 1669, *conſtat* D. Newtonum *fuiſſe, Amicum illum, de quo* D. Barrovius *in duabus prioribus Epiſtolis mentionem fecerat, quod his verbis conſignat.*

[†] I ſend you the Papers of my Friend I promis'd, which I preſume will give you much ſatisfaction; I pray, having peruſed them ſo much as you think good, remand them to me, according to his deſire when I ask'd him the Liberty to impart them to you; I pray give me notice of your receiving them, with your ſooneſt convenience, that I may be ſatisfied of their reception; becauſe I am afraid of them, venturing them by the Poſt, that I may not longer delay to correſpond with your deſire.

PRÆFATIO.

* Amici chartas tibi placuiſſe gaudeo; ‘eſt illi nomen *Newtonus*, Collegii noſtri Socius, & juvenis, (ſecundus enim, ex quo ‘Artium Magiſtri gradum cepit, jam agitur ‘annus,) et qui, eximio quo eſt acumine, ‘permagnos in hac re progreſſus fecit. Illas, ‘ſi vis, cum Nobili Domino Vicecomite ‘*Brounkero* communica.

Perſpecto jam D. Newtonum *hujus Tractatus Auctorem eſſe; ab eo ſciſcitatus ſum num penes ſe adhuc eſſet autographum, quod quidem ille exquirens invenit, & mihi tradidit, cum exemplari* Collinſiano *ad verbum uſque conveniens.*

Etiamſi vero hic Tractatus ad D. Collinſium *miſſus fuiſſet menſe* Julii 1669, *quod aliquantulum erat poſteaquam* D. Mercator *Logarithmotechniam ſuam in lucem ediderat; manifeſtum eſt ex quibuſdam aliis Epiſtolis, (quæ itidem inter* D. Collinſii *chartas fuerunt,) quod antea ſcriptus eſſet, imo quod* D. Newtonus *inveniſſet Methodum inveſtigandi Magnitudinum Dimenſiones per Infinitas Series vel aliquot annos antequam* D. Mercator *Librum ſuum in vulgus edidit; ut liquet ex Epiſtola a* Collinſio

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* I am glad my Friend's Paper gives you ſo much ſatisfaction, his Name is Mr. *Newton*, a Fellow of our College, and very young, (being but the ſecond Year Maſter of Arts,) but of an extraordinary Genius and Proficiency in theſe things; you may impart the Papers, if you pleaſe, to my Lord *Brounker*.

ad D. Jacobum Gregorium *data* 25. Novemb. 1669, *ubi hæc ſunt verba.*

* *Barrovius* Provinciam ſuam Publice prælegendi remiſit cuidam nomine *Newtono Cantabrigienſi*, quem tanquam Virum acutiſſimi ingenii, in Præfatione Prælectionum Opticarum memorat, & qui, antequam ederetur *Mercatoris* Logarithmotechnia, Methodum invenerat eandem, eamque ad omnes Curvas generaliter, & ad Circulum diverſimode applicârat.

Quinetiam D. Collinſius *in Epiſtola ad D.* Strode, *data* 26. Julii 1672, *ſic ſcribit.*

† Menſe *Septembri* 1668, *Mercator* Logarithmotechniam edidit ſuam, quæ ſpecimen hujus Methodi (i. e. Serierum Infinitarum) in unica tantum Figura, nempe, Quadratura Hyperbolæ continet. Haud multo poſtquam in publicum prodiret liber, exemplar unum Cl. *Walliſio Oxoniam* miſi, qui ſuum de eo judicium in *Actis Philoſophicis* ſtatim fecit: alterum *Barrovio Cantabrigiam*, qui quaſdam *Newtoni* chartas, qui jam *Barrovium* in Mathematicis Prælectionibus

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* Mr. *Barrow* hath reſign'd his Lecture's Place to one Mr. *Newton* of *Cambridge*, whom he mentions in his Optic Preface as a very ingenious Perſon; one who hath, before Mr. *Mercator's* Logarithmotechnia was extant, invented the ſame Method, and applied it generally to all Curves, and divers ways to the Circle.

‘onibus publicis excipit, extemplo remiſit: ‘Ex quibus & aliis, quæ olim ab Auctore ‘cum *Barrovio* communicata fuerant, patet ‘illam Methodum a dicto *Newtono* aliquot ‘annis ante excogitatam, & modo generali ‘applicatam fuiſſe: ita ut ejus ope in quavis ‘Figura Curvilinea propoſita, quæ una vel ‘pluribus Proprietatibus definitur, Quadratura vel Area dictæ Figuræ, accurata ſi ‘poſſibile ſit, ſin minus infinite propinqua; ‘Evolutio vel Longitudo lineæ Curvæ; Centrum gravitatis Figuræ; Solida ejus rotatione genita, & eorum Superficies; ſine ‘ulla Radicum Extractione obtineri queant.

Ubi obiter notemus in hac Collinſii *hiſtoriola, methodum argumentandi uſurpatam a* D. Newtono *in Tractatu ſuo* De Quadratura Curvarum, *quaſi digito monſtrari, dum dicit hanc Methodum exhibere Quadraturam Figurarum accuratam, ſi modo fieri poſſit, ſin minus in infinitum approximantem,*

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† In *September* 1668, Mr. *Mercator* publiſh'd his Logarithmotechnia, containing a Specimen of this Method, (that is, of Infinite Series) in one only Figure, to wit in the Quadrature of the Hyperbola. Not long after the Book came out, I ſent one of them to Dr. *Wallis* at *Oxford*, who forthwith gave his ſenſe of it in the Philoſ. Tranſactions: another of them I ſent to Dr. *Barrow* at *Cambridge*, who forthwith ſent me up ſome Papers of Mr. *Newton*, who is ſince become the Doctor's Succeſſor in the Mathematical Lectures there. By which, and former communications made thereof from the Author to the Doctor, it appears that the ſaid Method was invented ſome Years before by the ſaid Mr. *Newton*, and generally applied: ſo that thereby in any Curvilinear Figure propos'd, that is determin'd by one or more common Properties, by the ſame Method may be obtain'd the Quadrature or Area of the ſaid Figure, accurately when it can be done, but always infinitely near; the Evolution, or Length of the ſaid Curve Line; the Centre of Gravity of the Figure; its round Solids made by Rotation, and their Surfaces; and all perform'd without any Extraction of Roots.

tem, atque iſta omnia fieri ſine ulla Extractione Radicum; Hæc enim ipſa eſt argumentatio in dicto Tractatu obſervata: & propterea hanc Methodum ſaltem Anno 1672 *coætaneam extitiſſe non eſt dubitandum.*

Inveni etiam in exemplari Epiſtolæ, a D. Collinſio *ad* D. Davidem Gregorium *prædicti* Jacobi *fratrem, datæ* 11. Aug. 1676, *præter eadem fere quæ* D. Strode *ſcripſerat, etiam verba ſequentia:*

* Supradicta Serierum Infinitarum doctrina, a *Newtono* biennium ante excogitata fuit, quam ederetur *Mercatoris* Logarithmotechnia, & generaliter omnibus Figuris applicata.

Simulque affirmat ſe olim cum quibuſdam Academicis Pariſienſibus *hæc eadem ſcripto communicaſſe.*

Quapropter, cum D. Mercator *Librum ſuum Anno* 1668 *in lucem ediderit, ſequitur eandem Doctrinam Infinitarum Serierum* Figuris *omnibus generaliter applicatam fuiſſe Anno* 1666.

Denique in Epiſtola, circa idem tempus ad D. Oldenburgum *ſcripta, aſſerit* Collinſius, *Jacobum Gregorium* non niſi conſpecta aliqua e Seriebus *Newtoni*, quam illi impertierat, in eandem Serierum Methodum incidiſſe.

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* The ſaid Doctrine of Infinite Series was invented by Mr. *Newton*, about two Years before the Publication of Mr. *Mercator*'s Logarithmotechnia, and generally applied to all Curves.

Ex Newtoni *autem Schedis quibuſdam a me viſis intellexi, quod is Quadraturam Circuli, Hyperbolæ, & aliarum quarundam Curvarum per Series Infinitas ex* Walliſii *noſtri* Arithmetica Infinitorum, *per Interpolationem Serierum ejus, primo deduxit, idque Anno* 1665; *deinde Methodum excogitavit eaſdem Series per Diviſiones & Extractiones Radicum inveniendi, quam Anno ſequente generalem reddidit.*

Et cum ſcriptus fuerit hic Tractatus, quo tempore hæc recens inventa eſſent, ideo Cl. Auctor dignatus eſt multa in eo dilucidare, ad Reſolutionem Æquationum per Infinitas Series ſpectantia, quæ in aliis Libris fruſtra quæras.

His ſubjunximus diverſa Epiſtolarum Auctoris fragmenta, quæ ad eaſdem Doctrinas pertinent, quæq; olim inter Cl. Walliſii *Opera in lucem prodiere. Epiſtolas haud dedi integras, ut evitarem repetitionem non neceſſariam multarum rerum, quæ ſupra in Tractatu* De Analyſi per Æquationes Infinitas *traditæ ſunt: Quinimo Exempla quædam in iis Epiſtolis prætermiſi ipſius Cl. Auctoris monitu, Regulas ſuas per ſe ſatis claras eſſe credentis, neque ullam dilucidationem deſiderare.*

PRÆFATIO.

Inter D. Collinſii *ſchedas reperi autographum Epiſtolæ datæ* 8. Novemb. 1676, *cujus fragmentum ſub finem adjeci, & dignum luce putavi; quoniam in eo memoratur ſolutio Problematis admodum generalis in Comparatione Curvarum uſûs, quæ perficitur in Cor.* 2. *Prop.* 10. *Tract.* De Quadratura Curvarum: *Unde Lector intelligat ſolutionem illam Auctori jam tum innotuiſſe.*

Hiſce Fragmentis annexus eſt ille ipſe Tractatus De Quadratura Curvarum; *una cum altero* De Enumeratione Linearum Tertii Ordinis, *quorum uterque primum typis mandatus eſt Anno* 1704, *ad finem* Optices *eximiæ ejuſdem Cl Auctoris.*

Coronidis loco ſubjicitur Tractatulus, cui titulo eſt, Methodus Differentialis, *quem Cl. Auctoris permiſſu ex ejus autographo deſcripſi; Complectitur autem Doctrinam deſcribendi Curvas ex datis Differentiis differentiarum Ordinatarum. Hæc* Methodus Differentialis *innititur Problemati ducendi Curvam Parabolici generis per data quotcunque puncta; de quo Cl. Auctor olim mentionem fecerat in Epiſtola ſua ad D.*Oldenburgum 1676 *miſſa; & cujus ſolutionem dedit in Lem.* 5. *Lib.* 3. Princip. Philoſ. *non tamen prorſus eandem quoad Conſtructionem cum ea quam impræſentiarum tradimus.*

PRÆFATIO.

Hujus Geometriæ Newtonianæ *non minimam eſſe laudem duco, quod dum per limites* Rationum *Primarum & Ultimarum argumentatur, æque demonſtrationibus Apodicticis ac illa Veterum munitur; utpote quæ haud innititur duriuſculæ illi Hypotheſi quantitatum Infinite parvarum vel Indiviſibilium, quarum Evaneſcentia obſtat quominus eas tanquam quantitates ſpeculemur. Neque parum præcellere videtur, quod tam paucis innixa Propoſitionibus tam late ſeſe extendat hæc Matheſis, intra ſe omnia Problemata difficiliora vulgatas Methodos eludentia complectens; ſiquidem quicquid proponitur poterit Geometrice per alicujus Curvæ Aream effingi; vel per Methodum Univerſalem Extrahendi Radices ex Æquatione quavis erui; vel ad ſummum, ducendo Curvam per terminos quantitatum datarum ſolvi.*

W. Jones.

INDEX OPUSCULORUM

Quæ in hoc Libro continentur.

Pag.

De Analyſi per Æquationes Infinitas. 1

Fragmenta Epiſtolarum.

Ad D. Oldenburgum 13 Jun. *1676.* 23

Ad D. Oldenburgum 24 Octob. *1676.* 31

Ad D. Walliſium *Anno* 1692. 34

Ad D. Collinſium Nov. 8. *1676.* 38

De Quadratura Curvarum. 41

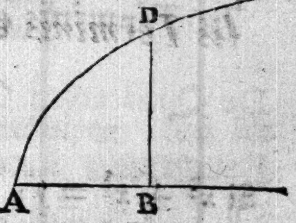
Enumeratio Linearum Tertii Ordinis. 69

Methodus Differentialis. 93

DE ANALYSI

Per Æquationes Numero Terminorum INFINITAS.

Methodum generalem, quam de Curvarum quantitate per Infinitam terminorum Seriem menſuranda, olim excogitaveram, in ſequentibus breviter explicatam potius quam accuratè demonſtratam habes.



BASI *AB* Curvæ alicujus *AD*, ſit Applicata *BD* perpendicularis: Et vocetur $AB = x$, $BD = y$, & ſint *a*, *b*, *c*, &c. Quantitates datæ, & *m*, *n*, Numeri Integri. Deinde,

Curvarum Simplicium Quadratura.

REGULA I.

Si $ax^{\frac{m}{n}} = y$; *Erit* $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = *Areæ ABD.*

Res Exemplo patebit.

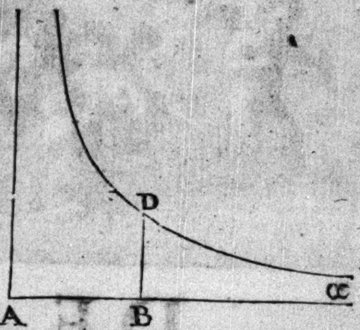
1. Si x^2 ($= 1x^{\frac{2}{1}}$) $= y$, hoc eſt, $a = 1 = n$, & $m = 2$; Erit $\frac{1}{3}x^3$ = ABD.

2. Si $4\sqrt{x}\ (=4x^{\frac{1}{2}})=y$; Erit $\frac{8}{3}x^{\frac{3}{2}}\ (=\frac{8}{3}\sqrt[2]{x^3})=ABD$.

3. Si $\sqrt[3]{x^5}\ (=x^{\frac{5}{3}})=y$; Erit $\frac{3}{8}x^{\frac{8}{3}}\ (=\frac{3}{8}\sqrt[3]{x^8})=ABD$.

4. Si $\frac{1}{x^2}\ (=x^{-2})=y$, id eſt, ſi $a=1=n$, & $m=-2$;

Erit $(\frac{1}{-1}x^{\frac{-1}{1}}=)\ -x^{-1}\ (=\frac{-1}{x})=\alpha BD$, infinite verſus α protenſæ, quam Calculus ponit negativam, propterea quod jacet ex altera parte Lineæ BD.



5. Si $\frac{1}{\sqrt{x^3}}\ (=x^{\frac{-3}{2}})=y$; Erit $(\frac{2}{-1}x^{\frac{-1}{2}}=)\ \frac{2}{-\sqrt{x}}=BD\alpha$.

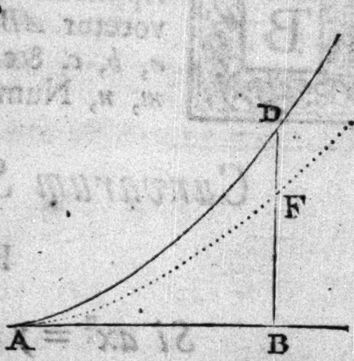
6. Si $\frac{1}{x}\ (=x^{-1})=y$; Erit $\frac{1}{0}x^{\frac{0}{1}}=\frac{1}{0}x^0=\frac{1}{0}\times 1=\frac{1}{0}=$ Infinitæ, qualis eſt Area Hyperbolæ ex utraque parte Lineæ BD.

Compoſitarum Curvarum Quadratura ex Simplicibus.

REGULA II.

Si valor ipſius* y *ex pluribus iſtiuſmodi Terminis componitur, Area etiam componetur ex Areis quæ a ſingulis Terminis emanant.

Exempla Prima.



Si $x^2+x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=ABD$.

Etenim ſi ſemper ſit $x^2=BF$, et $x^{\frac{3}{2}}=FD$, erit, ex præcedente Regula, $\frac{1}{3}x^3=$ ſuperficiei AFB deſcriptæ per Lineam BF, et $\frac{2}{5}x^{\frac{5}{2}}=$ AFD deſcriptæ per DF; Quare $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ toti ABD.

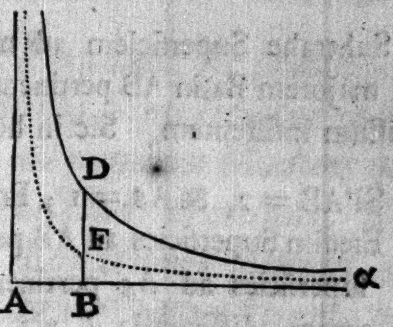
Sic ſi $x^2-x^{\frac{3}{2}}=y$; Erit $\frac{1}{3}x^3-\frac{2}{5}x^{\frac{5}{2}}=ABD$.

Et ſi $3x-2x^2+x^3-5x^4=y$; Erit $\frac{3}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4-x^5=ABD$.

Ex-

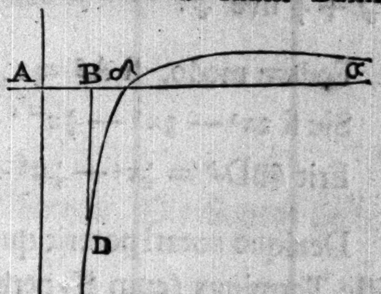
Exempla Secunda.

Si $x^{-2}+x^{-\frac{3}{2}}=y$; Erit $-x^{-1}-2x^{-\frac{1}{2}}=\alpha BD$. Vel ſi $x^{-2}-x^{-\frac{3}{2}}=y$; Erit $-x^{-1}+2x^{-\frac{1}{2}}=\alpha BD$.



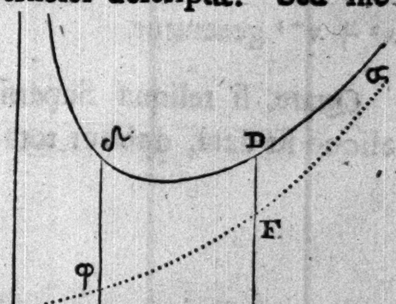
Quarum ſigna ſi mutaveris habebis Affirmativum valorem ($x^{-1}+2x^{-\frac{1}{2}}$ vel $x^{-1}-2x^{-\frac{1}{2}}$) ſuperficiei αBD, modo tota cadat ſupra baſim $AB\alpha$.

Sin aliqua pars cadat infra (quod fit cum Curva decuſſat ſuam Baſin inter B et α, ut hic vides in δ,) iſta parte a parte ſuperiori ſubducta, habebis valorem Differentiæ : Earum vero Summam ſi cupis, quære utramque Superficiem ſeorſim, et adde. Quod idem in reliquis hujus Regulæ exemplis notandum volo.



Exempla Tertia.

Si $x^{2}+x^{-2}=y$; Erit $\frac{1}{3}x^{3}-x^{-1}=$ Superficiei deſcriptæ. Sed hic notandum eſt, quod dictæ Superficei partes ſic inventæ jacent ex diverſo latere Lineæ BD.



Nempe, poſito $x^{2}=BF$, & $x^{-2}=FD$; Erit $\frac{1}{3}x^{3}=ABF$ Superficiei per BF deſcriptæ, & $-x^{-1}=DF\alpha$ Superficiei deſcriptæ per DF.

Et hoc ſemper accidit cum Indices $(\frac{m+n}{n})$ rationum Baſis x in valore Superficiei quæſitæ, ſint variis Signis affecti. In hujuſmodi Caſibus, pars aliqua $BD\delta\beta$ Superficiei media (quæ ſola dari poterit, cum Superficies ſit utrinque infinita) ſic invenitur.

Subtrahe Superficiem ad minorem Baſin $A\beta$ pertinentem, a Superficie ad majorem Baſin AB pertinente, & habebis $\beta BD\delta$ Superficiem differentiæ Baſium inſiſtentem. Sic in hoc Exemplo. (*Vide Fig. Præcedentem.*)

Si $AB = 2$, & $A\beta = 1$; Erit $\beta BD\delta = \frac{17}{6}$:

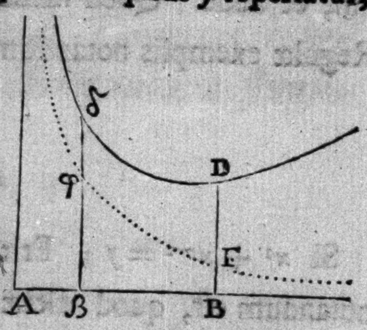
Etenim Superficies ad AB pertinens (viz. $ABF - DF\alpha$) erit $\frac{8}{3} - \frac{1}{2}$ ſive $\frac{13}{6}$; et ſuperficies ad $A\beta$ pertinens (viz. $A\varphi\beta - \delta\varphi\alpha$) erit $\frac{1}{3} - 1$, ſive $-\frac{2}{3}$: et earum differentia (viz. $ABF - DF\alpha - A\varphi\beta + \delta\varphi\alpha = \beta BD\delta$) erit $\frac{13}{6} + \frac{2}{3}$ ſive $\frac{17}{6}$.

Eodem modo, ſi $A\beta = 1$, $AB = x$; Erit $\beta BD\delta = \frac{2}{3} + \frac{1}{3}x^3 - x^{-1}$.

Sic ſi $2x^3 - 3x^5 - \frac{2}{3}x^{-4} + x^{-\frac{3}{5}} = y$, & $A\beta = 1$;

Erit $\beta BD\delta = \frac{1}{2}x^4 - \frac{1}{2}x^6 + \frac{2}{9}x^{-3} + \frac{5}{2}x^{\frac{2}{5}} - \frac{49}{18}$.

Denique notari poterit quod ſi quantitas x^{-1} in valore ipſius y reperiatur, iſte Terminus (cum Hyperbolicam ſuperficiem generat) ſeorſim a reliqius conſiderandus eſt.



Ut ſi $x^2 + x^{-3} + x^{-1} = y$: Sit $x^{-1} = BF$, & $x^2 + x^{-3} = FD$, ac $A\beta = 1$; Et erit $\delta\varphi FD = \frac{1}{6} + \frac{1}{3}x^3 - \frac{1}{2}x^{-2}$, utpote quæ ex Terminis $x^2 + x^{-3}$ generatur.

Quare, ſi reliqua Superficies $\beta\varphi FB$, quæ Hyperbolica eſt, ex Calculo aliquo ſit data, dabitur tota $\beta BD\delta$.

Aliarum

Aliarum Omnium Quadratura.

REGULA III.

Sin valor ipſius y, vel aliquis ejus Terminus ſit præcedentibus magis compoſitus, in Terminos Simpliciores reducendus eſt; operando in Literis ad eundem Modum quo Arithmetici in Numeris Decimalibus dividunt, Radices extrahunt, vel affectas Æquationes ſolvunt; & ex iſtis Terminis quæſitam Curvæ Superficiem, per præcedentes Regulas deinceps elicies.

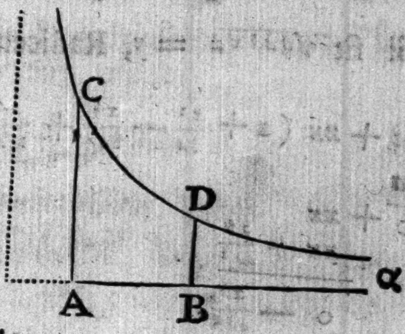
Exempla Dividendo.

Sit $\frac{aa}{b+x} = y$; Curva nempe exiſtente Hyperbola.

Jam ut Æquatio iſta a Denominatore ſuo liberetur, Diviſionem ſic inſtituo.

$$
b+x)\; aa + 0 \left(\frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4} \;\&c. \right.
$$

$$
\begin{array}{l}
\underline{aa + \frac{aax}{b}} \\
0 - \frac{aax}{b} + 0 \\
\quad \underline{- \frac{aax}{b} - \frac{aax^2}{b^2}} \\
\qquad 0 + \frac{aax^2}{b^2} + 0 \\
\qquad\quad \underline{+ \frac{aax^2}{b^2} + \frac{aax^3}{b^3}} \\
\qquad\qquad 0 - \frac{aax^3}{b^3} + 0 \\
\qquad\qquad\quad \underline{- \frac{aax^3}{b^3} - \frac{aax^4}{b^4}} \\
\qquad\qquad\qquad 0 + \frac{aax^4}{b^4} \\
\qquad\qquad\qquad\quad \&c.
\end{array}
$$



Et ſic vice hujus $y = \frac{aa}{b+x}$, nova prodit $y = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. ſerie iſtac infinite continuata; Adeoque (per Regulam Secundam) Area quæſita ABDC æqualis erit ipſi $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$ &c. infinitæ etiam ſeriei, cujus tamen Termini pauci initiales ſunt in uſum quemvis ſatis exacti, ſi modo x ſit aliquoties minor quam b.

Eodem modo, ſi ſit $\frac{1}{1+xx} = y$, Dividendo prodibit $y = 1 - x^2 + x^4 - x^6 + x^8$ &c. Unde (per Regulam Secundam) erit ABDC $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9$ &c.

Vel ſi Terminus xx ponatur in diviſore primus, hoc modo $xx + 1$), prodibit $x^{-2} - x^{-4} + x^{-6} - x^{-8}$ &c. pro valore ipſius y; Unde (per Regulam Secundam) erit BD∞ $= -x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7}$ &c. Priori modo procede cum x eſt ſatis parva, poſteriori cum ſatis magna ſupponitur.

Denique ſi $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$; Dividendo prodit

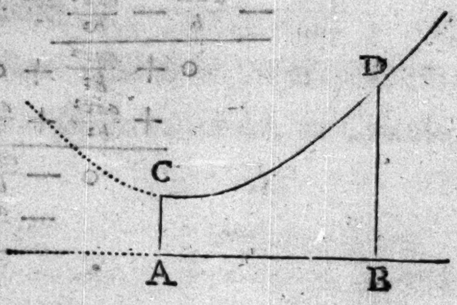
$2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$ &c. unde erit

ABDC $= \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3$ &c.

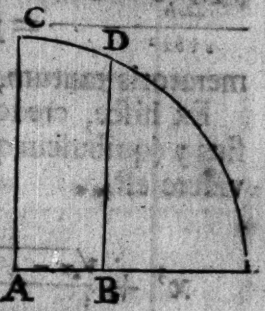
Exempla Radicem Extrahendo.

Si ſit $\sqrt{aa + xx} = y$, Radicem ſic extraho,

$$
\begin{array}{l}
aa + xx\ \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\right. \text{ \&c.} \\
\underline{aa} \\
0 + xx \\
\underline{\quad xx + \frac{x^4}{4a^2}} \\
\quad 0 - \frac{x^4}{4a^2} \\
\quad \underline{- \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}} \\
\qquad 0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
\qquad \underline{+ \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}}} \\
\qquad\quad 0 - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}} \\
\qquad\qquad \text{\&c.}
\end{array}
$$

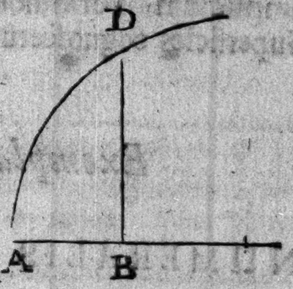


Unde, pro Æquatione $\sqrt{aa+xx}=y$, nova producitur, viz. $y=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ &c. Et (per Reg. 2.) Area quæſita ABDC erit $=ax+\frac{x^3}{6a}-\frac{x^5}{40a^3}+\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ &c. Et hæc eſt Quadratura Hyperbolæ.



Eodem modo, ſi ſit $\sqrt{aa-xx}=y$, ejus Radix erit $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$ &c. Adeoque Area quæſita ABDC erit æqualis $ax-\frac{x^3}{6a}-\frac{x^5}{40a^3}-\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$ &c. Et hæc eſt Quadratura Circuli.

Vel ſi ponas $\sqrt{x-xx}=y$; erit Radix æqualis infinitæ ſeriei $x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}$ &c. Et Area quæſita ABD æqualis erit $\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}-\frac{1}{72}x^{\frac{9}{2}}-\frac{5}{704}x^{\frac{11}{2}}$ &c. ſive $x^{\frac{1}{2}}$ in $\frac{2}{3}x-\frac{1}{5}x^2-\frac{1}{28}x^3-\frac{1}{72}x^4-\frac{5}{704}x^5$ &c. Et hæc eſt Areæ Circuli Quadratura.



Si $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}=y$, (Cujus Quadratura dat Longitudinem curvæ Ellipticæ;) Extrahendo radicem utramq; prodit

$$\frac{1+\frac{1}{2}ax^2-\frac{1}{8}a^2x^4+\frac{1}{16}a^3x^6-\frac{5}{128}a^4x^8}{1-\frac{1}{2}bx^2-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6-\frac{5}{128}b^4x^8}\ \&c.$$

Et Dividendo, ſicut fit in Fractionibus Decimalibus, habes

$$\begin{array}{lllll} 1 & +\frac{1}{2}bx^2 & +\frac{3}{8}b^2x^4 & +\frac{5}{16}b^3x^6 & +\frac{35}{128}b^4x^8\ \&c. \\ & +\frac{1}{2}a & +\frac{1}{4}ab & +\frac{3}{16}ab^2 & +\frac{5}{32}ab^3 \\ & & -\frac{1}{8}a^2 & -\frac{1}{16}a^2b & -\frac{3}{64}a^2b^2 \\ & & & +\frac{1}{16}a^3 & +\frac{1}{32}a^3b \\ & & & & -\frac{5}{128}a^4 \end{array}$$

Adeoque Aream quæſitam

$$\begin{array}{lll} x & +\frac{1}{6}bx^3 & +\frac{3}{40}b^2x^5\ \&c. \\ & +\frac{1}{6}a & +\frac{1}{20}ab \\ & & -\frac{1}{40}a^2 \end{array}$$

Sed obſervandum eſt, quod Operatio non raro abbreviatur per debitam Æquationis præparationem, ut in allato Exemplo $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}=y$. Si utramque partem fractionis per $\sqrt{1-bxx}$ multiplices prodibit $\frac{\sqrt{1\begin{smallmatrix}+a\\-b\end{smallmatrix}x^2-abx^4}}{1+bx^2}=y$, & reliquum opus perficitur extrahendo Radicem Numeratoris tantum, & dividendo per Denominatorem.

Ex hiſce, credo, ſatis patebit modus reducendi quemlibet valorem ipſius y (quibuſcunque Radicibus vel Denominatoribus ſit perplexus, ut hic videre eſt,

$$x^3+\frac{\sqrt{x-\sqrt{1-xx}}}{\sqrt[3]{ax^2+x^3}}-\frac{\sqrt[5]{x^3+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+x^2}-\sqrt[2]{2x-x^{\frac{2}{3}}}}=y)$$

in ſeries Infinitas ſimplicium Terminorum, ex quibus, per Regulam Secundam, quæſita Superficies cognoſcetur.

Exempla per Reſolutionem Æquationum,

NUMERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

Quia tota difficultas in Reſolutione latet, modum quo ego utor in Æquatione Numerali primum illuſtrabo.

Sit $y^3-2y-5=0$, reſolvenda: Et ſit 2, numerus qui minus quam decima ſui parte differt a Radice quæſita. Tum pono $2+p=y$, & ſubſtituo hunc ipſi valorem in Æquationem, & inde nova prodit $p^3+6p^2+10p-1=0$, cujus Radix p exquirenda eſt, ut quotienti addatur: Nempe (neglectis p^3+6p^2 ob parvitatem) $10p-1=0$, ſive $p=0{,}1$ prope veritatem eſt; itaque ſcribo 0,1 in quotiente, & ſuppono $0{,}1+q=p$, & hunc ejus valorem, ut prius ſubſtituo, unde prodit $q^3+6{,}3q^2+11{,}23q+0{,}061=0$.

Et cum $11{,}23q+0{,}061$ veritati prope accedit, ſive fere ſit q æqualis $-0{,}0054$ (dividendo nempe donec tot eliciantur Figuræ, quot locis primæ Figuræ hujus & principalis quotientis excluſive diſtant) ſcribo $-0{,}0054$ in inferiori parte quotientis, cum negativa ſit.

Et ſupponens $-0{,}0054+r=q$, hunc ut prius ſubſtituo, & operationem ſic produco quo uſq; placuerit. Verum ſi ad bis tot figuras tantum quot in Quotiente jam reperiuntur una dempta, operam continuare cupiam, pro q ſubſtituo $-0{,}0054+r$ in hanc $6{,}3q^2+11{,}23q+0{,}061$, ſcilicet primo ejus termino (q^3) propter exilitatem ſuam neglecto, & prodit $6{,}3r^2+11{,}16196r+0{,}000541708=0$ fere, ſive (rejecto $6{,}3r^2$) $r=\frac{-0{,}000541708}{11{,}16196}=-0{,}00004853$ fere, quam ſcribo in negativa parte Quotientis. Denique negativam partem Quotientis ab Affirmativa ſubducens habeo $2{,}09455147$ Quotientem quæſitam.

$y^3-2y-5=0$		$+2{,}10000000$ $-0{,}00544853$ $+2{,}09455147=y$
$2+p=y$	$+y^3$	$+8+12p+6p^2+p^3$
	$+2y$	$-4-2p$
	-5	-5
	Summa	$-1+10p+6p^2+p^3$
$0{,}1+q=p$	$+p^3$	$+0{,}001+0{,}03q+0{,}3q^2+q^3$
	$+6p^2$	$+0{,}06\quad+1{,}2\quad+6{,}0$
	$+10p$	$+1{,}\quad+10{,}$
	-1	$-1{,}$
	Summa	$+0{,}061+11{,}23q+6{,}3q^2+q^3$
$-0{,}0054+r=q$	$+6{,}3q^2$	$+0{,}0001183708-0{,}06804r+6{,}3r^2$
	$+11{,}23q$	$-0{,}060642\quad+11{,}23$
	$+0{,}061$	$+0{,}061$
	Summa	$+0{,}000541708+11{,}16196r+6{,}3r^2$
$-0{,}00004854+s=r$		

Æquationes plurium dimenſionum nihilo ſecius reſolvuntur, & operam ſub fine, ut hic factum fuit levabis, ſi primos ejus terminos gradatim omiſeris.

Præterea notandum eſt quod in hoc exemplo, ſi dubitarem an $0{,}1=p$ veritati ſatis accederet, pro $10p-1=0$, finxiſſem $6p^2+10p-1=0$, & ejus radicis primam figuram in Quotiente ſcripſiſſem; & ſecundam vel tertiam Quotientis figuram ſic explorare convenit, ubi in Æquatione iſta ultimo reſultante quadratum coefficientis penultimi termini, non ſit decies majus quam factus ex ultimo termino ducto in coefficientem termini antepenultimi.

Imo laborem plerumque minues præſertim in Æquationibus plurimarum dimenſionum, ſi figuras omnes Quotienti addendas dicto modo (hoc eſt extrahendo minorem radicum, ex tribus ultimis terminis Æquationis noviſſime reſultantis) exquiras: Iſto enim modo figuras duplo plures qualibet vice Quotienti lucraberis.

Hæc Methodus reſolvendi Æquationes pervulgata an ſit neſcio, certe mihi videtur præ reliquis ſimplex, & uſui accommodata. Demonſtratio ejus ex ipſo modo operandi patet, unde cum opus ſit, in memoriam facile revocatur.

Æquationes in quibus vel aliqui vel nulli Termini deſint, eadem fere facilitate tractantur; & Æquatio ſemper relinquitur, cujus Radix una cum acquiſita Quotiente adæquat Radicem Æquationis primo propoſitæ. Unde Examinatio Operis hic æque poterit inſtitui ac in reliqua Arithmetica, auferendo nempe Quotientem a Radice primæ Æquationis (ſicut Analyſtis notum eſt) ut Æquatio ultima vel Termini ejus duo treſve ultimi producantur inde. Quicquid laboris hic eſt, iſtud in Operatione ſubſtituendi quantitates unas pro aliis reperietur: Id quod varie perficias, at ſequentem modum maxime expeditum puto, præſertim ubi Numeri Coefficientes conſtant ex pluribus Figuris.

Sit $p+3$ ſubſtituenda pro y in hanc $y^4-4y^3+5y^2-12y+17=0$. Et cum iſta poſſit reſolvi in hanc formam $\overline{\overline{\overline{y-4}\times y+5}\times y-12}\times y+17=0$. Æquatio nova ſic generabitur $\overline{p-1}\times\overline{p+3}=p^2+2p-3$. et p^2+2p+2 in $p+3=p^3+5p^2+8p+6$. et p^3+5p^2+8p-6 in $p+3=p^4+8p^3+23p^2+18p-18$. et $p^4+8p^3+23p^2+18p-1=0$, quæ quærebatur.

LITERALIS ÆQUATIONUM AFFECTARUM RESOLUTIO.

His in numeris ſic oſtenſis: Sit Æquatio literalis $y^3+a^2y-2a^3+axy-x^3=0$, reſolvenda.

Primum inquiro valorem ipſius y cum x ſit nulla, hoc eſt, elicio Radicem hujus Æquationis $y^3+a^2y-2a^3=0$, & invenio eſſe $+a$. Itaque ſcribo $+a$ in Quotiente, & ſupponens $+a+p=y$, ſubſtituo pro y valorem ejus, & Terminos inde reſultantes ($p^3+3ap^2+4a^2p$, &c.) margini appono; Ex quibus aſſumo $+4a^2p+a^2x$ terminos utique ubi p & x ſeorſim ſunt minimarum dimenſionum, & eos nihilo fere æquales eſſe ſuppono; ſive $p=-\frac{1}{4}x$ fere, vel $p=-\frac{1}{4}x+q$. Et ſcribens

$-\frac{1}{4}x$ in Quotiente, ſubſtituo $-\frac{1}{4}x+q$ pro p; Et terminos inde reſultantes iterum in margine ſcribo, ut vides in annexo ſchemate, & inde aſſumo Quantitates $+4a^2q-\frac{1}{16}ax^2$, in quibus utiq; q & x ſeorſim ſunt minimarum dimenſionum, & fingo $q=\frac{xx}{64a}$ fere, ſive $q=+\frac{xx}{64a}+r$; & adnectens $+\frac{xx}{64a}$ Quotienti, ſubſtituo $\frac{xx}{64a}+r$ pro q; & ſic procedo quo uſque placuerit.

$$y^3+a^2y-2a^3+axy-x^3=0$$

$$y=a-\frac{x}{4}+\frac{x^2}{64a}-\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}\ \&c.$$

$+a+p=y$	$+y^3$	$+a^3+3a^2p+3ap^2+p^3$
	$+a^2y$	$+a^3+a^2p$
	$+axy$	$+a^2x+axp$
	$-2a^3$	$-2a^3$
	$-x^3$	$-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$	$-\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$
	$+3ap^2$	$+\frac{3}{16}ax^2-\frac{3}{2}axq+3aq^2$
	$+4a^2p$	$-a^2x+4a^2q$
	$+axp$	$-\frac{1}{4}ax^2+axq$
	$+a^2x$	$+a^2x$
	$-x^3$	$-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$	$+\frac{3x^4}{4096a}+\frac{3}{32}x^2r+3ar^2$
	$+4a^2q$	$+\frac{1}{16}ax^2+4a^2r$
	$-\frac{1}{2}axq$	$-\frac{1}{128}x^3-\frac{1}{2}axr$
	$+\frac{3}{16}x^2q$	$-\frac{3x^4}{1024a}+\frac{3}{16}x^2r$
	$-\frac{1}{16}ax^2$	$-\frac{1}{16}ax^2$
	$-\frac{65}{64}x^3$	$-\frac{65}{64}x^3$

$$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\ (+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$$

Sin duplo tantum plures Quotienti terminos, uno dempto, jungendos adhuc vellem: Primo termino (q^3) Æquationis noviſſime reſultantis miſſo, & iſta etiam parte ($-\frac{3}{4}xq^2$) ſecundi, ubi x eſt tot dimenſionum quot in penultimo termino Quotientis; In reliquos terminos ($3aq^2+4a^2q$, *&c.*)

margini adſcriptos ut vides, ſubſtituo $\frac{x^2}{64a} + r$ pro q; & ex ultimis duobus terminis ($\frac{15x^4}{4096a} - \frac{131}{128}x^3 + \frac{9}{32}x^2r - \frac{1}{2}axr + 4a^2r$) Æquationis inde reſultantis, facta diviſione $4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{131}{128}x^3 - \frac{15x^4}{4096a}$ (elicio $+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ Quotienti adnectendos.

Denique Quotiens iſta ($a - \frac{x}{4} + \frac{xx}{64a}$, &c.) per Regulam ſecundam, dabit $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3}$, &c. pro Area quæſita, quæ ad veritatem tanto magis accedit, quanto x ſit minor.

Alius modus eaſdem Reſolvendi.

Sin valor Areæ tanto magis ad veritatem accedere debet quanto x ſit major; Exemplum eſto $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Itaque hanc reſoluturus excerpo terminos $y^3 + x^2y - 2x^3$ in quibus x & y vel ſeorſim, vel ſimul multiplicatæ, ſunt & plurimarum, & æqualium ubique dimenſionum; & ex iis quaſi nihilo æqualibus Radicem elicio. Hanc invenio eſſe x, & in Quotiente ſcribo. Vel quod eodem recidit, ex $y^3 + y - 2$ (unitate pro x ſubſtituta) Radicem extraho quæ hic prodit 1, & eam per x multiplico, & factum (x) in Quotiente ſcribo. Denique pono $x + p = y$, & ſic procedo ut in priori Exemplo, donec habeam Quotientem $x - \frac{a}{4} + \frac{aa}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$, &c. adeoque Aream $\frac{x^2}{2} - \frac{ax}{4}$ $+ \boxed{\frac{aa}{64x}} - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$, de qua vide exempla tertia Regulæ ſecundæ. Lucis gratia dedi hoc exemplum in omnibus idem cum priori, modo x & a ſibi invicem ibi ſubſtituantur, ut non opus eſſet aliud Reſolutionis exemplum hic adjungere.

Area autem ($\frac{xx}{2} - \frac{ax}{4} + \boxed{\frac{aa}{64x}}$ &c. terminatur ad Curvam quæ juxta Aſymptoton aliquam in infinitum ſerpit; & Termini initiales ($x - \frac{1}{4}a$) valoris extracti de y, in Aſymptoton iſtam ſemper terminantur; unde portionem Aſymptoti facile invenies. Idem ſemper notandum eſt cum Area deſignatur terminis plus pluſque diviſis per x continue, præterquam quod vice Aſymptoti rectæ quandoque habeatur Parabola Conica, vel alia magis compoſita.

Sed hunc modum miſſum faciens, utpote particularem, quia non applicabilem Curvis in orbem ad inſtar Ellipſium flexis; de altero modo per exemplum $y^3+a^2y+axy-2a^3-x^3=0$, ſupra oſtenſo (ſcilicet quo demenſiones ipſius x in numeratoribus quotientis perpetuo augeantur) annotabo ſequentia.

1. Si quando accidit quod valor ipſius y, cum x nullum eſſe fingitur, ſit quantitas ſurda vel penitus ignota, licebit illam litera aliqua deſignare. Ut in exemplo, $y^3+a^2y+axy-2a^3-x^3=0$, ſi radix hujus $y^3+a^2y-2a^3$ fuiſſet ſurda vel ignota, finxiſſem quamlibet (b) pro ea ponendam; et reſolutionem ut ſequitur perfeciſſem. Scribens b in Quotiente, ſuppono $b+p=y$, & iſtum pro y ſubſtituo, ut vides; unde nova p^3+3bp^2, &c. reſultat, rejectis terminis $b^3+a^2b-2a^3$, qui nihilo ſunt æquales, propterea quod b ſupponitur Radix hujus $y^3+a^2y-2a^3=0$. Deinde termini $3b^2p+a^2p+abx$ dant $-\frac{abx}{3b^2+a^2}$ quotienti apponendum, et $-\frac{abx}{3b^2+a^2}+q$ ſubſtituendum pro p, &c.

$$y^3+aay+axy-2a^3-x^3=0. \quad \text{Sit } cc=3b^2+a^2.$$

$$y=b-\frac{abx}{c^2}+\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}-\frac{a^5bx^3}{c^8}+\frac{a^5b^3x^3}{c^{10}} \text{ \&c.}$$

$b+p=y$	y^3	$+b^3+3b^2p+3bp^2+p^3$
	$+axy$	$+abx+axp$
	$+aay$	$+aab+aap$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$\frac{-abx}{cc}+q=p$	p^3	$-\frac{a^3b^3x^3}{c^6}$ &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4}-\frac{6ab^2x}{c^2}q$ &c.
	$+axp$	$-\frac{a^2bx^2}{c^2}+axq$
	$+ccp$	$-abx+ccq$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$$c^2+ax-\frac{6ab^2x}{cc}\Big)\frac{a^4bx^2}{c^4}+x^3+\frac{a^3b^3x^3}{c^6}\Big(\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8} \text{ \&c.}$$

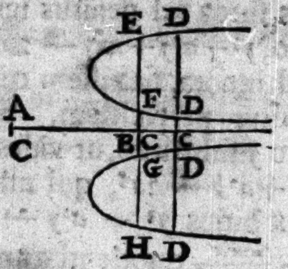
Completo opere, ſumo numerum aliquem pro a, & hanc $y^3+a^2y-2a^3=0$, ſicut de numerali æquatione oſtenſum ſupra reſolvo; & radicem ejus pro b ſubſtituo.

2. Si dictus valor ſit nihil, hoc eſt ſi in æquatione reſolvenda nullus ſit terminus niſi qui per x vel y ſit multiplicatus, ut in hac $y^3-axy+x^3=0$; tum terminos $(-axy+x^3)$ ſeligo in quibus x ſeorſim & y etiam ſeorſim ſi fieri poteſt, alias per x multiplicata, ſit minimarnm dimenſionum. Et

illi dant $+\frac{xx}{a}$ pro primo termino quotientis, & $\frac{xx}{a}+p$ pro y ſubſtituendum. In hac $y^3-a^2y+axy-x^3=0$, licebit primum terminum quotientis vel ex $-a^2y-x^3$, vel ex y^3-a^2y elicere.

3. Si valor iſte ſit imaginarius, ut in hac $y^4+y^2-2y+6-x^2y^2-2x+x^2+x^4=0$, augeo vel imminuo quantitatem x donec dictus valor evadat realis.

Sic in annexo ſchemate, cum AC (x) nulla eſt, tum CD (y) eſt imaginaria.



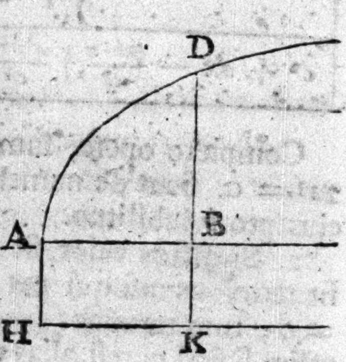
Sin minuatur AC per datam AB, ut BC fiat x; tum poſito quod BC (x) ſit nulla, CD (y) erit valore quadruplici (CE, CF, CG, vel CH) realis; quarum radicum (CE, CF, CG, vel CH) quælibet poteſt eſſe primus terminus quotientis, prout ſuperficies BEDC, BFDC, BGDC, vel BHDC deſideratur. In aliis etiam caſibus, ſi quando hæſitas, te hoc modo extricabis.

Deniq; ſi index poteſtatis ipſius x vel y ſit fractio, reduco ipſum ad integrum: ut in hoc exemplo $y^3-xy^{\frac{1}{2}}+x^{\frac{4}{3}}=0$. Poſito $y^{\frac{1}{2}}=v$, & $x^{\frac{1}{3}}=z$, reſultabit $v^6-z^3v+z^4=0$, cujus radix eſt $v=z+z^3$, &c. ſive (reſtituendo valores) $y^{\frac{1}{2}}=x^{\frac{1}{3}}+x$, &c. & quadrando $y=x^{\frac{2}{3}}+2x^{\frac{4}{3}}$, &c.

Et hæc de areis curvarum inveſtigandis dicta ſufficiant. Imo cum Problemata omnia de curvarum Longitudine, de quantitate & ſuperficie ſolidorum, deque Centro Gravitatis, poſſunt eo tandem reduci ut quæratur quantitas Superficiei planæ linea curva terminatæ, non opus eſt quicquam de iis adjungere. In iſtis autem quo ego operor modo dicam breviſſime.

Applicatio prædictorum ad reliqua iſtiuſmodi Problemata.

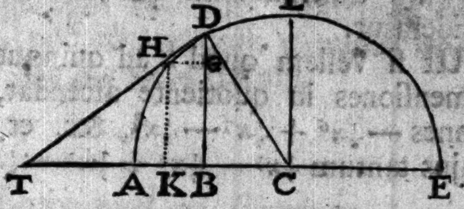
Sit ABD curva quævis, & AHKB rectangulum cujus latus AH vel BK eſt unitas. Et cogita rectam DBK uniformiter ab AH motam, areas ABD & AK deſcribere; & quod BK (1) ſit momentum quo AK (x) & BD (y) momentum quo ABD gradatim augetur; & quod ex momento BD perpetim dato, poſſis, per prædictas regulas, aream ABD ipſo deſcriptam inveſtigare, ſive cum AK (x) momento 1 deſcripta conferre.



Jam qua ratione Superficies ABD ex momento ſuo perpetim dato, per præcedentes regulas elicitur, eadem quælibet alia quantitas ex momento ſuo ſic dato elicietur. Exemplo res fiet clarior.

Longitudines Curvarum invenire.

Sit ADLE circulus cujus arcus AD longitudo eſt indaganda. Ducto tangente DHT, & completo indefinite parvo rectangulo HGBK, & poſito AE $= 1 = 2$AC. Erit ut BK ſive GH, momentum Baſis AB (x), ad HD momentum Arcus AD :: BT : DT :: BD ($\sqrt{x-xx}$) : DC ($\frac{1}{2}$) :: 1 (BK) : $\frac{1}{2\sqrt{x-xx}}$ (DH). Adeoque $\frac{1}{2\sqrt{x-xx}}$ ſive $\frac{\sqrt{x-xx}}{2x-2xx}$ eſt momentum Arcus AD.



Quod reductum fit $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ &c. Quare, per regulam ſecundam, longitudo Arcus AD eſt

$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}$ &c.

ſive $x^{\frac{1}{2}}$in $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5$, &c.

Non ſecus ponendo CB eſſe x, & radium CA eſſe 1, invenies Arcum LD eſſe $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

Sed notandum eſt quod unitas iſta quæ pro momento ponitur eſt Superficies cum de Solidis, & linea cum de ſuperficiebus, & punctum cum de lineis (ut in hoc exemplo) agitur.

Nec vereor loqui de unitate in punctis, ſive lineis infinite parvis, ſi quidem proportiones ibi jam contemplantur Geometræ, dum utuntur methodis Indiviſibilium.

Ex his fiat conjectura de ſuperficiebus & quantitatibus ſolidorum, ac de Centris Gravitatum.

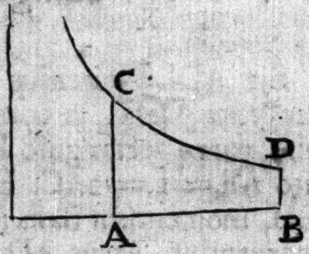
Invenire prædictorum converſum.

Verum ſi e contra ex area vel longitudine &c. Curvæ alicujus data longitudo Baſis AB deſideratur, ex æquationibus per præcedentes regulas inventis extrahatur radix de x.

Inven-

Inventio Baſis ex Area data.

Ut ſi ex area ABDC Hyperbolæ ($\frac{1}{1+x} = y$) data, cupiam baſim AB inveſtigare, area iſta z nominata, extraho radicem hujus z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. neglectis illis terminis in quibus x eſt plurium dimenſionum quam z in quotiente deſideratur.



Ut ſi vellem quod z ad quinque tantum dimenſiones in quotiente aſcendat, negligo omnes $-\frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, &c. et radicem hujus tantum $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$ extraho.

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 \text{ \&c.}$$

$z + p = x$	$+\frac{1}{5}x^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}x^4$	$-\frac{1}{6}z^4 - z^3p$, &c.
	$+\frac{1}{3}x^3$	$+\frac{1}{3}z^3 + z^2p + zp^2$, &c.
	$-\frac{1}{2}x^2$	$-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+\ x$	$+z + p$
	$-\ z$	$-z$
$\frac{1}{2}z^2 + q = p$	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4 - \frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{2}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4 + z^2q$
	$-z\ p$	$-\frac{1}{2}z^3 - zq$
	$+\ \ p$	$+\frac{1}{2}z^2 + q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$

$$1 - z + \tfrac{1}{2}z^2\,)\ \tfrac{1}{6}z^3 - \tfrac{1}{8}z^4 + \tfrac{1}{20}z^5\ (\ \tfrac{1}{6}z^3 + \tfrac{1}{24}z^4 + \tfrac{1}{120}z^5$$

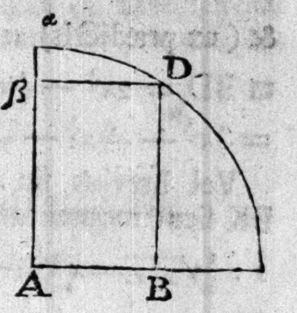
Analyſin ut vides exhibui propter adnotanda duo ſequentia.

1. Quod inter ſubſtituendum, iſtos terminos ſemper omitto quos nulli deinceps uſui fore prævideam. Cujus rei regula eſto, quod poſt primum terminum ex qualibet quantitate ſibi collaterali reſultantem non addo plures terminos dextrorſum quam iſtius primi termini index dimenſionis

ab indice dimenſionis maximæ unitatibus diſtat. Ut in hoc exemplo, ubi maxima dimenſio eſt 5, omiſi omnes terminos poſt z^5, poſt z^4 poſui unicum, & duos tantum poſt z^3. Cum radix extrahenda (x) ſit parium ubique, vel imparium dimenſionum, hæc eſto regula; Quod poſt primum terminum ex qualibet quantitate ſibi collaterali reſultantem non addo plures terminos dextrorſum, quam iſtius primi termini index dimenſionis ab indice dimenſionis maximæ binis unitatibus diſtat; vel ternis unitatibus, ſi indices dimenſionum ipſius x unitatibus ubique ternis a ſe invicem diſtant, & ſic de reliquis.

2. Cum video p, q, vel r, &c. in æquatione noviſſime reſultante eſſe unius tantum dimenſionis, ejus valorem, hoc eſt, reliquos terminos quotienti addendos, per diviſionem quæro. Ut hic vides factum.

Inventio Baſis ex data Longitudine Curvæ.



Si ex dato arcu αD Sinus AB deſideratur; æquationis $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. ſupra inventæ, (poſito nempe AB $= x$, αD $= z$, & Aα $= 1$,) radix extracta erit $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

Et præterea ſi Coſinum Aβ ex iſto arcu dato cupis, fac Aβ $(= \sqrt{1-zz}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

De Serie progreſſionum continuanda.

Hic obiter notetur, quod 5 vel 6 terminis iſtarum radicum cognitis, eas plerumque ex analogia obſervata poteris ad arbitrium producere.

Sic hanc $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. produces dividendo ultimum terminum per hos ordine numeros 2, 3, 4, 5, 6, 7, &c.

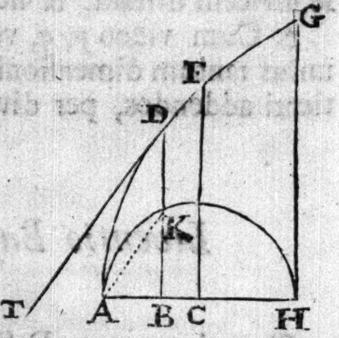
Et hanc $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, &c. per hos 2×3, 4×5, 6×7, 8×9, 10×11, &c.

Et hanc $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. per hos 1×2, 3×4, 5×6, 7×8, 9×10, &c.

Et hanc $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. multiplicando per hos $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, &c. Et ſic in reliquis.

Applicatio prædictorum ad Curvas Mechanicas.

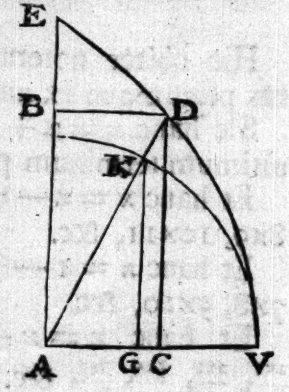
Et hæc de curvis Geometricis dicta ſufficiant. Quinetiam curva etiamſi Mechanica ſit, methodum tamen noſtram nequaquam reſpuit.



Exemplo ſit Trochoides, ADFG, cujus vertex A, & axis AH, & AKH rota qua deſcribitur. Et quæratur Superficies ABD. Jam poſito AB $= x$, BD $= y$, ut ſupra, & AH $= 1$; primo quæro Longitudinem ipſius BD. Nempe ex natura Trochoidis eſt KD $=$ arcui AK. Quare tota BD $=$ BK $+$ arc. AK. Sed eſt BK ($= \sqrt{x - xx}$) $= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. & (ex prædictis) arcus AK $= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$, &c. Ergo tota BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. Et (per Reg. 2.) area ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}}$, &c.

Vel brevius ſic: Cum recta AK tangenti TD parallela ſit, erit AB ad BK ſicut momentum lineæ AB ad momentum lineæ BD, hoc eſt $x : \sqrt{x - xx} :: 1 : \frac{1}{x}\sqrt{x - xx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$, &c. Quare (per Reg. 2.) BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$, &c. Et ſuperficies ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}}$, &c.

Non diſſimili modo (poſito C centro circuli, & CB $= x$) obtinebis aream CBDF, &c.



Sit area ABDV Quadratricis VDE (cujus vertex eſt V, & A centrum circuli interioris VK cui aptatur) invenienda. Ducta qualibet AKD, demitto perpendiculares DB, DC, KG. Eritque KG : AG :: AB (x) : BD (y), ſive $\frac{x \times \mathrm{AG}}{\mathrm{KG}} = y$. Verum ex natura Quadratricis eſt BA ($=$ DC) $=$ arcui VK, ſive VK $= x$. Quare poſito AV $= 1$, erit GK $= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. ex ſupra oſtenſis, & GA $= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$, &c.

Adeoque $y\ (=\frac{x\times AG}{KG})=\frac{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6}{1-\frac{1}{6}x^2+\frac{1}{120}x^4-\frac{1}{5040}x^6}$ &c. ſive, diviſione facta, $y=1-\frac{1}{3}x^2-\frac{1}{45}x^4-\frac{2}{945}x^6$, &c. et (per Reg. 2.) area $AVDB=x-\frac{1}{9}x^3-\frac{1}{225}x^5-\frac{2}{6615}x^7$, &c.

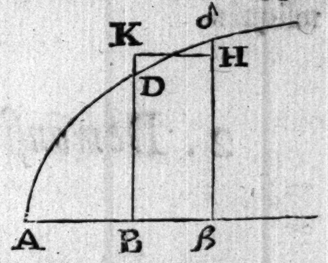
Sic longitudo Quadratricis VD, licet calculo difficiliori, determinabilis eſt. Nec quicquam hujuſmodi ſcio ad quod hæc methodus idque variis modis, ſeſe non extendit. Imo tangentes ad curvas Mechanicas (ſi quando id non alias fiat) hujus ope ducantur. Et quicquid vulgaris Analyſis per æquationes ex finito terminorum numero conſtantes (quando id ſit poſſibile) perficit, hæc per æquationes infinitas ſemper perficiat: Ut nil dubitaverim nomen Analyſis etiam huic tribuere. Ratiocinia nempe in hac non minus certa ſunt quam in illa, nec æquationes minus exactæ; licet omnes earum terminos, nos homines & rationis finitæ nec deſignare neque ita concipere poſſimus, ut quantitates inde deſideratas exacte cognoſcamus: Sicut radices ſurdæ finitarum æquationum nec numeris nec quavis arte Analytica ita poſſunt exhiberi ut alicujus quantitas a reliquis diſtincta exacte cognoſcatur.

Denique ad Analyticam merito pertinere cenſeatur cujus beneficio curvarum areæ & longitudines &c. (id modo fiat) exacte & Geometrice determinentur. Sed iſta narrandi non eſt locus, reſpicienti duo præ reliquis demonſtranda occurrunt.

1. *Demonſtratio quadraturæ curvarum ſimplicium in Regula prima.*

Præparatio pro Regula prima demonſtranda.

Sit itaque curvæ alicujus $AD\delta$ Baſis $AB=x$, perpendiculariter applicata $BD=y$, & area $ABD=z$, ut prius. Item ſit $B\beta=o$, $BK=v$, & rectangulum $B\beta HK$ (ov) æquale ſpatio $B\beta\delta D$.



Eſt ergo $A\beta=x+o$, & $A\delta\beta=z+ov$. His præmiſſis, ex relatione inter x & z ad arbitrium aſſumpta quæro y iſto, quem ſequentem vides, modo.

Pro lubitu ſumatur $\frac{2}{3}x^{\frac{3}{2}}=z$, ſive $\frac{4}{9}x^3=zz$. Tum $x+o$ $(A\beta)$ pro x, & $z+ov$ $(A\delta\beta)$ pro z ſubſtitutis, prodibit $\frac{4}{9}$ in $x^3+3x^2o+3xo^2+o^3=$ (ex natura curvæ) $z^2+2zov+o^2v^2$.

Et ſublatis ($\frac{4}{9}x^3$ & zz) æqualibus, reliquiſque per o diviſis, reſtat $\frac{4}{9}$ in $3x^2 + 3xo + o^2 = 2zv + ov^2$. Si jam ſupponamus $B\beta$ in infinitum diminui & evaneſcere, ſive o eſſe nihil, erunt v & y æquales, & termini per o multiplicati evaneſcent, quare reſtabit $\frac{4}{9} \times 3xx = 2zv$, ſive $\frac{2}{3}xx$ $(= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, ſive $x^{\frac{1}{2}}$ $(= \frac{x^2}{x^{\frac{3}{2}}}) = y$. Quare e contra ſi $x^{\frac{1}{2}} = y$, erit $\frac{2}{3}x^{\frac{3}{2}} = z$.

Demonſtratio.

Vel generaliter, ſi $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; ſive, ponendo $\frac{na}{m+n} = c$, & $m+n = p$, ſi $cx^{\frac{p}{n}} = z$, ſive $c^n x^p = z^n$: tum $x + o$ pro x, & $z + ov$ (ſive, quod perinde eſt, $z + oy$) pro z, ſubſtitutis, prodit c^n in $x^p + pox^{p-1}$, &c. $= z^n + noyz^{n-1}$, &c. reliquis nempe terminis, qui tandem evaneſcerent, omiſſis. Jam ſublatis $c^n x^p$ & z^n æqualibus, reliquiſque per o diviſis, reſtat $c^n px^{p-1} = nyz^{n-1}$ $(= \frac{nyz^n}{z} = \frac{nyc^n x^p}{cx^{\frac{p}{n}}}$ ſive, dividendo per $c^n x^p$, erit $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$ ſive $pcx^{\frac{p-n}{n}} = ny$; vel reſtituendo $\frac{na}{m+n}$ pro c, & $m+n$ pro p, hoc eſt, m pro $p-n$, & na pro pc, fiet $ax^{\frac{m}{n}} = y$. Quare e contra, ſi $ax^{\frac{m}{n}} = y$, erit $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$. Q. E. D.

Inventio curvarum quæ poſſunt quadrari.

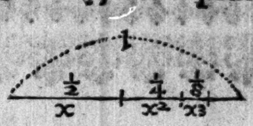
Hinc in tranſitu notetur modus quo curvæ tot quot placuerit, quarum areæ ſunt cognitæ, poſſunt inveniri; ſumendo nempe quamlibet æquationem pro relatione inter aream z & baſin x ut inde quæratur applicata y. Ut ſi ſupponas $\sqrt{aa+xx} = z$, ex calculo invenies $\frac{x}{\sqrt{aa+xx}} = y$. Et ſic de reliquis.

2. *Demonſtratio reſolutionis æquationum affectarum.*

Alterum demonſtrandum eſt literalis æquationum affectarum reſolutio. Nempe quod Quotiens, cum x ſit ſatis parva, quo magis producitur eo magis ad veritatem accedit, ut defectus (p, q, vel r, &c.) quo diſtat ab exacto valore ipſius y, tandem evadat minor quavis data quantitate; & in infinitum producta ſit ipſi y æqualis. Quod ſic patebit.

1. Quoniam ex ultimo termino æquationum quarum p, q, r, &c. ſunt radices, quantitas illa in qua x eſt minimæ dimenſionis (hoc eſt, pluſquam dimidium iſtius ultimi termini, ſi ſupponis x ſatis parvam eſſe) in qualibet operatione perpetuo tollitur: iſte ultimus terminus (per 1. 10. *Elem.*) tandem evadet minor quavis data quantitate; & prorſus evaneſcet ſi opus infinite continuatur.

Nempe ſi $x = \frac{1}{2}$, erit x dimidium omnium $x + x^2 + x^3 + x^4$, &c. et x^2 dimidium omnium $x^2 + x^3 + x^4 + x^5$, &c. Itaque ſi $x \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x + x^2 + x^3$, &c. et x^2 pluſquam dimidium omnium $x^2 + x^3 + x^4$, &c. Sic ſi $\frac{x}{b} \sqsupset \frac{1}{2}$, erit x pluſquam dimidium omnium $x + \frac{x^2}{b} + \frac{x^3}{bb}$, &c. Et ſic de reliquis. Et numeros coefficientes quod attinet, illi plerumque decreſcunt perpetuo, vel ſi quando increſcant, tantum opus eſt ut x aliquoties adhuc minor ſupponatur.



2. Si ultimus terminus alicujus æquationis continuo diminuatur donec tandem evaneſcat, una ex ejus radicibus etiam diminuetur donec cum ultimo termino ſimul evaneſcat.

3. Quare quantitatum p, q, r, &c. unus valor continuo decreſcit donec tandem, cum opus in infinitum producitur, penitus evaneſcat.

4. Sed valores iſtarum p, q, vel r, &c. una cum quotiente eatenus extracta adæquant radices æquationis propoſitæ (Sic in reſolutione æquationis $y^3 + auy + axy - 2a^3 - x^3 = 0$, ſupra oſtenſa, percipies $y = a + p = a - \frac{1}{4}x + q = a - \frac{1}{4}x + \frac{xx}{64a} + r$, &c.) Unde ſatis liquet propoſitum quod quotiens infinite producta eſt una ex valoribus de y.

Idem patebit ſubſtituendo quotientem pro y in æquationem propoſitam. Videbis enim terminos illos ſeſe perpetuo deſtruere in quibus x eſt minimarum dimenſionum.

CUM in Epiſtolis D. Newtoni, *vel in lucem jamdudum editis, vel quæ in manus noſtras inciderunt, reperiantur aliqua quæ ad hanc Doctrinam pertinent, ea excerpere & huic Tractatui adjungere viſum eſt.*

EXCERPTA
Ex Epiſtolis D. NEWTONI
Ad Methodum
FLUXIONUM,
ET
SERIERUM INFINITARUM
Spectantibus.

Fragmentum **Epiſtolæ ad D.* Oldenburgium 13 Junii 1676 *miſſæ.*

FRactiones in Infinitas Series reducuntur per diviſionem; & quantitates radicales per extractionem radicum, perinde inſtituendo operationes iſtas in ſpeciebus ac inſtitui ſolent in decimalibus numeris. Hæc ſunt fundamenta harum reductionum; ſed extractiones radicum, multum abbreviantur per hoc *Theorema.*

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Ubi $P+PQ$ ſignificat quantitatem cujus Radix, vel etiam dimenſio quævis, vel radix dimenſionis, inveſtiganda eſt. P, primum terminum quantitatis ejus; Q, reliquos terminos diviſos per primum. Et $\frac{m}{n}$, numeralem indicem dimenſionis ipſius $P+PQ$: Sive dimenſio illa integra ſit; ſive (ut ita loquar) fracta; ſive affirmativa, ſive negativa. Nam, ſicut Analyſtæ, pro aa, aaa, &c. ſcribere ſolent a^2, a^3, &c. ſic ego, pro $\surd a$, $\surd a^3$, $\surd c. a^5$, &c. ſcribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, ſcribo a^{-1}, a^{-2}, a^{-3}.

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* *Extat Epiſtola in Tom.* 3. *Operum* Walliſii.

Et ſic pro $\frac{aa}{\sqrt{c:\overline{a^3+b^2x}}}$ ſcribo $aa\times\overline{a^3+b^2x}|^{-\frac{1}{3}}$; & pro $\frac{a^2b}{\sqrt{c:\overline{a^3+b^2x}\times\overline{a^3+b^2x}}}$ ſcribo $a^2b\times\overline{a^3+b^2x}|^{-\frac{2}{3}}$: In quo ultimo caſu, ſi $\overline{a^3+b^2x}|^{-\frac{2}{3}}$ concipiatur eſſe $\overline{P+PQ}|^{\frac{m}{n}}$ in Regula; erit $P=a^3$, $Q=\frac{b^2x}{a^3}$, $m=-2$, & $n=3$. Denique, pro terminis inter operandum inventis in quoto, uſurpo A, B, C, D, &c. nempe A pro primo termino $P^{\frac{m}{n}}$, B pro ſecundo $\frac{m}{n}AQ$, & ſic deinceps. Cæterum uſus Regulæ patebit exemplis.

Exempl. 1. Eſt $\sqrt{c^2+x^2}$ (ſeu $\overline{c^2+x^2}|^{\frac{1}{2}}$) $= c+\frac{x^2}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}+\frac{7x^{10}}{256c^9}$ &c. Nam, in hoc caſu, eſt $P=c^2$, $Q=\frac{x^2}{c^2}$, $m=1$, $n=2$, A $(=P^{\frac{m}{n}}=\overline{cc}|^{\frac{1}{2}})=c$, B$(=\frac{m}{n}AQ)=\frac{x^2}{2c}$, C $(=\frac{m-n}{2n}BQ)=-\frac{x^4}{8c^3}$, & ſic deinceps.

Exempl. 2. Eſt $\sqrt{5:\overline{c^5+c^4x-x^5}}$ (i. e. $\overline{c^5+c^4x-x^5}|^{\frac{1}{5}}$) $=c+\frac{c^4x-x^5}{5c^4}-\frac{2c^8x^2+4c^4x^6-2x^{10}}{25c^9}+$ &c. Ut patebit ſubſtituendo in allatam Regulam, 1 pro m, 5 pro n, c^5 pro P, & $\frac{c^4x-x^5}{c^5}$ pro Q. Poteſt etiam $-x^5$ ſubſtitui pro P, & $\frac{c^4x+c^5}{-x^5}$ pro Q, et tunc evadet $\sqrt{5:\overline{c^5+c^4x-x^5}}=-x+\frac{c^4x+c^5}{5x^4}+\frac{2c^8x^2+4c^9x+c^{10}}{25x^9}+$ &c. Prior modus eligendus eſt, ſi x valde parvum ſit; poſterior, ſi valde magnum.

Exempl. 3. Eſt $\frac{N}{\sqrt{3:\overline{y^3-a^2y}}}$ (hoc eſt, $N\times\overline{y^3-a^2y}|^{-\frac{1}{3}}$) æqualis $\overline{N\times\frac{1}{y}+\frac{a^2}{3y^3}+\frac{2a^4}{9y^5}+\frac{14a^6}{81y^7}+\text{\&c.}}$ Nam $P=y^3$, $Q=\frac{-aa}{yy}$, $m=-1$, $n=3$. A $(=P^{\frac{m}{n}}=y^{3\times-\frac{1}{3}})=y^{-1}$, hoc eſt $\frac{1}{y}$. B$(=\frac{m}{n}AQ=-\frac{1}{3}\times\frac{1}{y}\times\frac{-aa}{yy})=\frac{aa}{3y^3}$, &c.

Exempl. 4. Radix cubica ex quadrato-quadrato ipſius $d+e$, (hoc eſt, $\overline{d+e}|^{\frac{4}{3}}$) eſt $d^{\frac{4}{3}}+\frac{4ed^{\frac{1}{3}}}{3}+\frac{2e^2}{9d^{\frac{2}{3}}}-\frac{4e^3}{81d^{\frac{5}{3}}}+$ &c.
Nam $P=d$, $Q=\frac{e}{d}$, $m=4$, $n=3$, A $(=P^{\frac{m}{n}})=d^{\frac{4}{3}}$, &c.

Exempl. 5. Eodem modo ſimplices etiam poteſtates eliciuntur. Ut ſi quadrato-cubus ipſius $d+e$, (hoc eſt, $\overline{d+e}|^5$, ſeu $\overline{d+e}|^{\frac{5}{1}}$) deſideretur: erit, juxta Regulam, $P=d$, $Q=\frac{e}{d}$, $m=5$, & $n=1$; adeoque A$(=P^{\frac{m}{n}})=d^5$, B $(=\frac{m}{n}AQ)=5d^4e$, & ſic $C=10d^3e^2$, $D=10d^2c^3$, $E=5de^4$, $F=e^5$, & G $(=\frac{m-5n}{6n}FQ)=0$. Hoc eſt, $\overline{d+e}|^5=d^5+5d^4e+10d^3e^2+10d^2e^3+5de^4+e^5$.

Exem.

Exempl. 6. Quinetiam Diviſio, ſive ſimplex ſit, ſive repetita, per eandem Regulam perficitur. Ut ſi $\frac{1}{d+e}$ (hoc eſt $\overline{d+e}|^{-1}$ ſive $\overline{d+e}|^{-\frac{1}{1}}$) in ſeriem ſimplicium terminorum reſolvendum ſit: Erit juxta Regulam $P=d$, $Q=\frac{e}{d}$, $m=-1$, $n=1$, & $A(=P^{\frac{m}{n}}=d^{-\frac{1}{1}})=d^{-1}$ ſeu $\frac{1}{d}$, $B(=\frac{m}{n}AQ=-1\times\frac{1}{d}\times\frac{e}{d}=-\frac{e}{d^2}$, & ſic $C=\frac{ee}{d^3}$, $D=-\frac{e^3}{d^4}$, &c. Hoc eſt $\frac{1}{d+e}=\frac{1}{d}-\frac{e}{d^2}+\frac{e^2}{d^3}-\frac{e^3}{d^4}+$ &c.

Exempl. 7. Sic et $\overline{d+e}|^{-3}$ (hoc eſt unitas ter diviſa per $d+e$, vel ſemel per cubum ejus,) evadit $\frac{1}{d^3}-\frac{3e}{d^4}+\frac{6e^2}{d^5}-\frac{10e^3}{d^6}+$ &c.

Exempl. 8. Et $N\times\overline{d+e}|^{-\frac{1}{3}}$, (hoc eſt N diviſum per radicem cubicam ipſius $d+e$,) evadit $N\times\overline{\frac{1}{d^{\frac{1}{3}}}-\frac{e}{3d^{\frac{4}{3}}}+\frac{2e^2}{9d^{\frac{7}{3}}}-\frac{14e^3}{81d^{\frac{10}{3}}}}+$ &c.

Exempl. 9. Et $N\times\overline{d+e}|^{-\frac{3}{5}}$ (hoc eſt N diviſum per radicem quadrato-cubicam ex cubo ipſius $d+e$, ſive $\frac{N}{\sqrt{5:\overline{d^3+3d^2e+3de^2+e^3}}}$) evadit $N\times\frac{1}{d^{\frac{3}{5}}}-\frac{3e}{5d^{\frac{8}{5}}}+\frac{12e^2}{25d^{\frac{13}{5}}}-\frac{52e^3}{125d^{\frac{18}{5}}}+$ &c.

Per eandem Regulam Geneſis Poteſtatum, Diviſiones per Poteſtates aut per quantitates radicales, & Extractiones radicum altiorum in numeris etiam commode inſtituuntur.

Extractiones Radicum affectarum in ſpeciebus imitantur earum extractiones in numeris, ſed methodus *Vietæ* & *Oughtredi* noſtri huic negotio minus idonea eſt: Quapropter aliam excogitare adactus ſum. [*Hujus ſpecimen exhibetur in Tractatu præcedente Pag.* 8.]

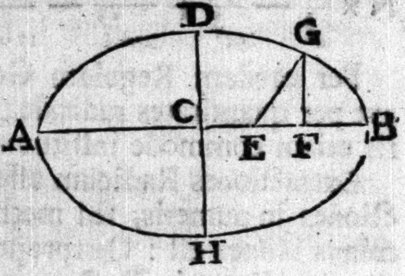
Quomodo ex æquationibus, ſic ad infinitas ſeries reductis, Areæ & Longitudines curvarum, Contenta & Superficies ſolidorum, vel quorumlibet ſegmentorum figurarum quarumvis, eorumque Centra gravitatis determinantur; & quomodo etiam Curvæ omnes Mechanicæ ad ejuſmodi æquationes infinitarum ſerierum reduci poſſint, indeque Problemata circa illas reſolvi perinde ac ſi Geometricæ eſſent; nimis longum foret deſcribere, ſufficiat ſpecimina quædam talium Problematum recenſuiſſe: Inque iis, brevitatis gratia, literas A, B, C, D, &c. pro terminis ſeriei, ſicut ſub initio, nonnunquam uſurpabo.

1. *Si ex dato ſinu recto, vel ſinu verſo, Arcus deſideretur*: Sit radius r, & ſinus rectus x: Eritque Arcus $=x+\frac{x^3}{6r^2}+\frac{3x^5}{40r^4}+\frac{5x^7}{112r^6}$ &c. hoc eſt, $=x+\frac{1\times1\times xx}{2\times3\times rr}A+\frac{3\times3xx}{4\times5rr}B+\frac{5\times5xx}{6\times7rr}C+\frac{7\times7xx}{8\times9rr}D+$ &c.

Vel, ſit d diameter, ac x ſinus verſus; & erit Arcus æqualis $d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c. hoc eſt, $= \sqrt{dx}$ in $1 + \frac{x}{6d} + \frac{3x^2}{40d^2} + \frac{5x^3}{112d^3} +$ &c.

2. Si viciſſim, *ex dato Arcu deſideretur ſinus*: Sit radius r, & arcus z: Eritque ſinus rectus $= z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{362880r^8} -$ &c. hoc eſt, $= z - \frac{zz}{2 \times 3rr}A - \frac{zz}{4 \times 5rr}B - \frac{zz}{6 \times 7rr}C -$ &c. Et ſinus verſus $= \frac{z^2}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{40320r^7} +$ &c. hoc eſt, $= \frac{zz}{1 \times 2r} - \frac{zz}{3 \times 4rr}A - \frac{zz}{5 \times 6rr}B - \frac{zz}{7 \times 8rr}C -$ &c.

3. *Si Arcus capiendus ſit in ratione data ad alium Arcum*: Eſto diameter $= d$, chorda arcus dati $= x$, & arcus quæſitus ad arcum illum datum ut n ad 1; Eritque arcus quæſiti Chorda $= nx + \frac{1-nn}{2 \times 3dd}xxA + \frac{9-nn}{4 \times 5dd}xxB + \frac{25-nn}{6 \times 7dd}xxC + \frac{36-nn}{8 \times 9dd}xxD + \frac{49-nn}{10 \times 11dd}xxE +$ &c. Ubi nota, quod cum n eſt numerus impar, ſeries deſinet eſſe infinita, & evadet eadem quæ prodit per Vulgarem Algebram ad multiplicandum datum angulum per iſtum numerum n.



4. Si in Axe alterutro AB, Ellipſeos ADB (cujus centrum C, & axis alter DH) detur punctum aliquod E, circa quod recta EG occurrens Ellipſi in G, motu angulari feretur; & ex data area ſectoris Elliptici BEG, quæratur recta GF, quæ a puncto G ad axem AB normaliter demittitur: Eſto BC $= q$, DC $= r$, EB $= t$, ac duplum areæ BEG $= z$; et erit GF $= \frac{1}{t}z - \frac{q}{6r^2t^4}z^3 + \frac{10q^2 - 9qt}{120r^4t^7}z^5 - \frac{280q^3 + 504q^2t - 225qt^2}{5040r^6t^{10}}z^7 +$ &c. Sic itaque Aſtronomicum illud *Kepleri* Problema reſolvi poteſt.

5. In eadem Ellipſi, ſi ſtatuatur CD $= r$, $\frac{CB^2}{CD} = c$, & CF $= x$: Erit arcus

$$
\begin{array}{llllll}
\text{Ellipticus DG} = x + \frac{1}{6c^2}x^3 & + \frac{1}{10rc^3}x^5 & + \frac{1}{14r^2c^4}x^7 & + \frac{1}{18r^3c^5}x^9 & + \frac{1}{22r^4c^6}x^{11} & + \text{\&c.} \\
 & - \frac{1}{40c^4} & - \frac{1}{28rc^5} & - \frac{1}{24r^2c^6} & - \frac{1}{22r^3c^7} & \\
 & & + \frac{1}{112c^6} & + \frac{1}{48rc^7} & + \frac{3}{88r^2c^8} & \\
 & & & - \frac{5}{1152c^8} & - \frac{5}{352rc^9} & \\
 & & & & + \frac{7}{2816c^{10}} &
\end{array}
$$

Hic numerales Coefficientes ſupremorum terminorum ($\frac{1}{6}, \frac{1}{10}, \frac{1}{14}$, &c.) ſunt in Muſica progreſſione: Et numerales Coefficientes omnium inferiorum in unaquaque columna prodeunt multiplicando continuo numeralem Coefficientem ſupremi termini per terminos hujus progreſſionis

$\frac{\frac{1}{2}n-1}{2}$, $\frac{\frac{3}{3}n-3}{4}$, $\frac{\frac{5}{4}n-5}{6}$, $\frac{\frac{7}{5}n-7}{8}$, $\frac{\frac{9}{6}n-9}{10}$, &c. Ubi n ſignificat numerum dimenſionum ipſius c in denominatore iſtius ſupremi termini. E. *g.* ut terminorum infra $\frac{1}{22r^4c^6}$, numerales coefficientes inveniantur, pono $n=6$, ducoque $\frac{1}{22}$ (numeralem coefficientem ipſius $\frac{1}{22r^4c^6}$) in $\frac{\frac{1}{2}n-1}{2}$, hoc eſt, in 1; & prodit $\frac{1}{22}$, numeralis coefficiens termini proxime inferioris: dein duco hunc $\frac{1}{22}$ in $\frac{\frac{3}{3}n-3}{4}$, ſive in $\frac{n-3}{4}$, hoc eſt, in $\frac{3}{4}$; & prodit $\frac{3}{88}$ numeralis coefficiens tertii termini in iſta columna. Atque iſta $\frac{3}{88}\times\frac{\frac{5}{4}n-5}{6}$ facit $\frac{5}{352}$ numeralem coefficientem quarti termini; & $\frac{5}{352}\times\frac{\frac{7}{5}n-7}{8}$ facit $\frac{7}{2816}$ numeralem coefficientem infimi termini. Idem in aliis ad infinitum uſque columnis præſtari poteſt: Adeoque valor ipſius DG per hanc Regulam pro lubitu produci.

Adhæc, ſi BF dicatur x, ſitque r latus rectum Ellipſeos, & $e=\frac{r}{AB}$; Erit Arcus Ellipticus

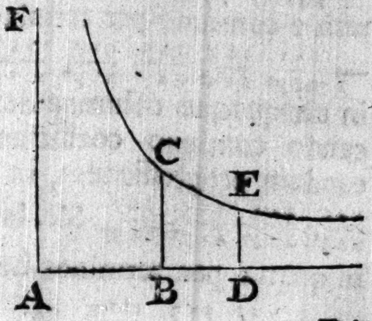
$$BG = \sqrt{rx} \text{ in } \overline{1 \left.\frac{\begin{array}{l}+2\\-\frac{3}{2}e\end{array}}{3r}\right\}x \left.\frac{\begin{array}{l}-2\\+3e\\-\frac{5}{8}e^2\end{array}}{5r^2}\right\}x^2 \left.\frac{\begin{array}{l}+4\\-9e\\+\frac{23}{4}e^2\\-\frac{7}{16}e^3\end{array}}{7r^3}\right\}x^3 \left.\frac{\begin{array}{l}-10\\+30e\\-\frac{123}{4}e^2\\+\frac{91}{8}e^3\\-\frac{45}{128}e^4\end{array}}{9r^4}\right\}x^4, + \&c.}$$

Quare, ſi ambitus totius Ellipſeos deſideretur; Biſeca CB in F, & quære Arcum DG, per prius Theorema, & Arcum BG per poſterius.

6. Si, vice verſa, *ex dato arcu Elliptico* DG, *quæratur Sinus ejus* CF; tum dicto CD $=r$, $\frac{CB^2}{CD}=c$, & arcu illo DG $=z$; Erit

$$CF = z - \frac{1}{6c^2}z^3 \begin{array}{l}-\frac{1}{10rc^3}\\+\frac{13}{120c^4}\end{array} z^5 \begin{array}{l}-\frac{1}{14r^2c^4}\\+\frac{71}{420rc^5}\\-\frac{493}{5040c^6}\end{array} z^7 - \&c.$$

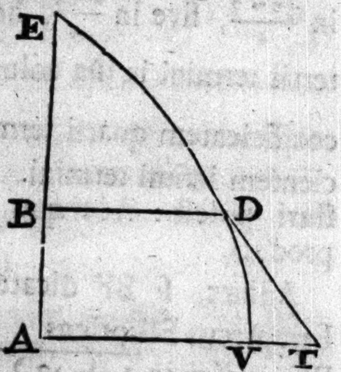
Quæ autem de Ellipſi dicta ſunt, omnia facile accommodantur ad Hyperbolam; mutatis tantum ſignis ipſorum c & e ubi ſunt imparium dimenſionum.



7. Præterea, ſi ſit CE Hyperbola, cujus Aſymptoti AD, AF rectum angulum FAD conſtituant; & ad AD erigantur utcunque perpendicula BC, DE occurrentia Hyperbolæ in C & E: & AB dicatur a, BC b, & area BCED z;

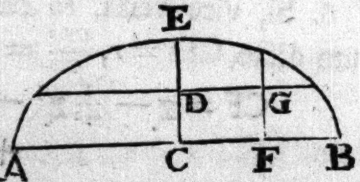
Erit BD $= \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} +$ &c. Ubi coefficientes denominatorum prodeunt multiplicando terminos hujus Arithmeticæ progreſſionis, 1, 2, 3, 4, 5, &c. in ſe continuo. Et hinc ex Logarithmo dato poteſt numerus ei competens inveniri.

8. Eſto VDE *Quadratrix*, cujus vertex eſt V, exiſtente A centro & AE ſemi-diametro Circuli ad quem aptatur, & angulo VAE recto: Demiſſoque ad AE perpendiculo quovis DB, & acta Quadratricis Tangente DT occurrente axi ejus AV in T: Dic AV $= a$, & AB $= x$; Eritq; DB $= a - \frac{x^2}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5} -$ &c. Et VT $= \frac{x^2}{3a} + \frac{x^4}{15a^3} + \frac{2x^6}{189a^5} +$ &c. Et Area AVDB $= ax - \frac{x^3}{9a} - \frac{x^5}{225a^3} - \frac{2x^7}{6615a^5} -$ &c. Et Arcus VD $= x + \frac{2x^3}{27a^2} + \frac{14x^5}{2025a^4} + \frac{604x^7}{893025a^6} +$ &c. Unde viciſſim, ex dato BD, vel VT, aut area AVDB, arcuve VD, per reſolutionem affectarum æquationum erui poteſt x ſeu AB.



9. Eſto denique AEB *Sphæroides*, revolutione Ellipſeos AEB circa axem AB genita, & ſecta planis quatuor, AB per axem tranſeunte, DG parallelo AB, CDE perpendiculariter biſecante axem, & FG parallelo CE: ſitque recta CB $= a$, CE $= c$, CF $= x$, & FG $= y$. Et Sphæroideos ſegmentum CDGF dictis quatuor planis comprehenſum, erit

$$+ 2cxy - \frac{x}{3c}y^3 - \frac{x}{20c^3}y^5 - \frac{x}{56c^5}y^7 - \frac{5x}{576c^7}y^9 - \text{\&c.}$$
$$- \frac{cx^3}{3a^2} - \frac{x^3}{18ca^2} - \frac{x^3}{40c^3a^2} - \frac{5x^3}{336c^5a^2} - \text{\&c.}$$
$$- \frac{cx^5}{20a^4} - \frac{x^5}{40ca^4} - \frac{3x^5}{160c^3a^4} - \text{\&c.}$$
$$- \frac{cx^7}{56a^6} - \frac{5x^7}{336ca^6} - \text{\&c.}$$
$$- \frac{5cx^9}{576a^8} - \text{\&c.}$$
$$- \text{\&c.}$$



Ubi numerales coefficientes ſupremorum terminorum ($2, -\frac{1}{3}, -\frac{1}{20}, -\frac{1}{56}, -\frac{5}{576}$, &c.) in infinitum producuntur multiplicando primum coeffitem 2 continuo per terminos hujus progreſſionis $-\frac{1\times1}{2\times3}, \frac{1\times3}{4\times5}, \frac{3\times5}{6\times7}, \frac{5\times7}{8\times9}, \frac{7\times9}{10\times11}$, &c. Et numerales coefficientes terminorum in unaquaque columna deſcendentium in infinitum producuntur multiplicando continuo coefficientem ſupremi termini in prima columna per eandem progreſſionem, in ſecunda autem per terminos hujus $\frac{1\times1}{2\times3}, \frac{3\times3}{4\times5}, \frac{5\times5}{6\times7}, \frac{7\times7}{8\times9}$, &c. in tertia per terminos hujus $\frac{3\times1}{2\times3}, \frac{5\times3}{4\times5}, \frac{7\times5}{6\times7}, \frac{9\times7}{8\times9}$, &c. in quarta per terminos hujus $\frac{5\times1}{2\times3}, \frac{7\times3}{4\times5}, \frac{9\times5}{6\times7}$, &c. in quinta per terminos hujus $\frac{7\times1}{2\times3}, \frac{9\times3}{4\times5}, \frac{11\times5}{6\times7}$, &c. Et ſic in infinitum.

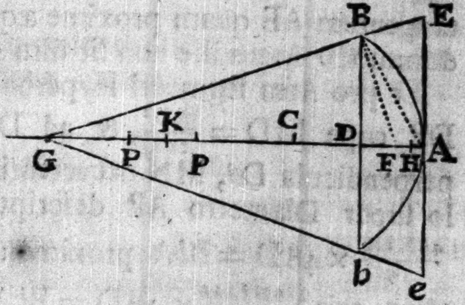
Et eodem modo ſegmenta aliorum ſolidorum deſignari, & valores eorum aliquando commode per ſeries quaſdem numerales in infinitum produci poſſunt.

Ex his videre eſt, quantum fines Analyſeos per hujuſmodi infinitas æquationes amplientur: Quippe quæ, earum beneficio, ad omnia pæne dixerim problemata (ſi numeralia *Diophanti* & ſimilia excipias) ſeſe extendit.

Non tamen omnino univerſalis evadit, niſi per ulteriores quaſdam methodos eliciendi ſeries infinitas. Sunt enim quædam Problemata in quibus non liceat ad ſeries infinitas per diviſionem vel extractionem radicum ſimplicium affectarumve, pervenire. Sed quomodo in iſtis caſibus procedendum ſit, jam non vacat dicere; ut neque alia quædam tradere quæ circa reductionem infinitarum ſerierum in finitas, ubi rei natura tulerit, excogitavi. Nam parcius ſcribo, quod hæ ſpeculationes diu mihi faſtidio eſſe cœperint, adeo ut ab iiſdem jam per quinque fere annos abſtinuerim.

Unum tamen addam: quod poſtquam Problema aliquod ad infinitam æquationem deducitur, poſſint inde variæ approximationes in uſum Mechanicæ, nullo fere negotio formari; quæ, per alias methodos quæſitæ, multo labore temporiſque diſpendio conſtare ſolent.

Cujus rei exemplo eſſe poſſunt Tractatus *Hugenii* aliorumque de Quadratura circuli. Nam, ut ex data arcus chorda A, & dimidii arcus chorda B, arcum illum proxime aſſequaris; finge arcum illum eſſe z, & circuli radium r; juxtaque ſuperiora erit A (nempe duplum ſinus dimidii z) $= z - \frac{z^3}{4 \times 6r^2} + \frac{z^5}{4 \times 4 \times 120r^4} - \&c.$ Et $B = \frac{1}{2}z - \frac{z^3}{2 \times 16 \times 6r^2} + \frac{z^5}{2 \times 16 \times 16 \times 120r^4} - \&c.$ Duc jam B in numerum fictitium n, & a producto aufer A, & reſidui ſecundum terminum (nempe $- \frac{nz^3}{2 \times 16 \times 6r^2} + \frac{z^3}{4 \times 6r^2},$) eo ut evaneſcat, pone $= 0$; indeque emerget $n = 8$, et erit $8B - A = 3z * - \frac{3z^5}{64 \times 120r^4} + \&c.$ hoc eſt $\frac{8B - A}{3} = z$; errore tantum exiſtente $\frac{z^5}{7680r^4} - \&c.$ in exceſſu. Quod eſt Theorema *Hugenianum*.



Inſuper, ſi in Arcus B*b*, ſagitta AD indefinite producta, quæratur punctum G, a quo actæ rectæ GB, G*b* abſcindant Tangentem E*e* quam proxime æqualem Arcui iſti: Eſto circuli centrum C, diameter $AK = d$, & ſagitta $AD = x$: Et erit DB $(= \sqrt{dx - x^2})$

$$= d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}} - \&c.$$

Et AE (= AB) $= d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} + \&c.$

Et AE — DB : AD : : AE : AG ; Quare $AG = \frac{3}{2}d - \frac{1}{5}x - \frac{12x^2}{175d}$ — vel + &c. Finge ergo $AG = \frac{3}{2}d - \frac{1}{5}x$; et viciſſim erit DG $(\frac{3}{2}d - \frac{6}{5}x)$: DB : : DA : AE — DB. Quare $AE - DB = \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}}$ + &c. Adde DB ; & prodit

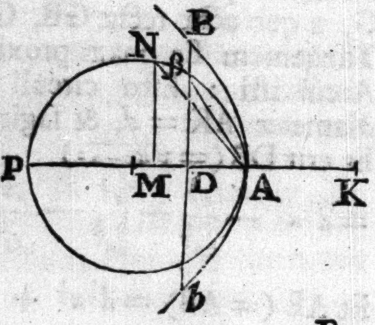
$$AE = d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}} + \&c.$$

Hoc aufer de valore ipſius AE ſupra habito, & reſtabit error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}}$ + vel — &c. Quare in AG, cape AH quintam partem DA, et KG = HC, et actæ GBE, G*be* abſcindent Tangentem E*e* quam proxime æqualem arcui BA*b* ; errore tantum exiſtente $\frac{16x^3}{525d^3}\sqrt{dx}$ + vel — &c. multo minore ſcilicet quam in Theoremate *Hugenii*. Quod ſi fiat 7 AK : 3 AH : : DH : *n* ; et capiatur KG = CH — *n*, erit error adhuc multo minor.

Atque ita, ſi Circuli ſegmentum aliquod BA*b* per Mechanicam deſignandum eſſet : Primo reducerem Aream iſtam in Infinitam ſeriem, puta hanc $BbA = \frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}}$ — &c. Dein quærerem conſtructiones Mechanicas quibus hanc ſeriem proxime aſſequerer ; cujuſmodi ſunt hæ : Age rectam AB, et erit ſegmentum $BbA = \overline{\frac{2}{3}AB + BD} \times \frac{4}{5}AD$ proxime ; exiſtente ſcilicet errore tantum $\frac{x^3}{70d^2}\sqrt{dx}$ + &c. in defectu : Vel proximius, erit ſegmentum illud (biſecto AD in F, & acta recta BF) $= \frac{4BF + AB}{15} \times 4AD$; exiſtente errore ſolummodo $\frac{x^3}{560d^2}\sqrt{dx}$ + &c. qui ſemper minor erit quam $\frac{1}{1500}$ totius ſegmenti, etiamſi ſegmentum illud ad uſque ſemicirculum augeatur.

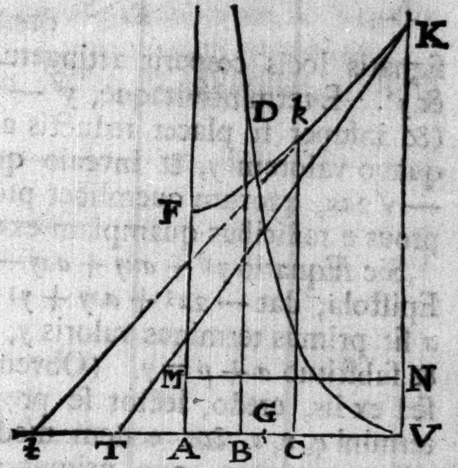
Sic et in Ellipſi BA*b*, [*Vid. Fig. Præcedent.*] cujus vertex A, axis alteruter AK, & latus rectum AP ; cape $PG = \frac{1}{2}AP + \frac{19AK - 21AP}{10AK} \times AP$. In Hyperbola vero, cape $PG = \frac{1}{2}AP + \frac{19AK + 21AP}{10AK} \times AP$. Et acta recta GBE abſcindet tangentem AE quam proxime æqualem arcui Elliptico vel Hyperbolico AB, dummodo Arcus ille non ſit nimis magnus.

Et pro Area ſegmenti Hyperbolici B*b*A ; in DP cape $MD = \frac{3AD^2}{4AK}$, & ad D & M erige perpendicula Dβ, MN occurrentia ſemicirculo ſuper Diametro AP deſcripto : Eritque $\frac{4AN + A\beta}{15} \times 4AD = BbA$ proxime : Vel proximius, erit $\frac{21AN + 4A\beta}{75} \times 4AD = BbA$; ſi modo capiatur $DM = \frac{5AD^2}{7AK}$.



Fragmentum Epiſtolæ D. Newtoni, *ad* D. Oldenburgium 24 Octob. 1676 *miſſæ.*

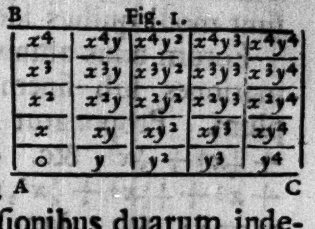
Longitudo *Ciſſoidis* ſic conſtruitur. Sit VD *Ciſſois*, AV Diameter Circuli ad quem aptatur, V vertex, AF Aſymptota ejus, ac DB perpendiculare quodvis ad AV demiſſum. Cum ſemi-axe AF = AV, & ſemi-parametro AG = $\frac{1}{3}$AV, deſcribatur Hyperbola F*k*K; & inter AB & AV ſumpta AC media proportionali, erigantur ad C & V perpendicula C*k*, VK Hyperbolæ occurrentia in *k* & K; Et agantur rectæ KT, *kt* tangentes Hyperbolam in eiſdem K & *k*, et occurrentes AV in T & *t*; Et ad AV conſtituatur rectangulum AVNM æquale ſpatio TK*kt*. Et *Ciſſoidis* VD longitudo erit Sextupla altitudinis VN. Demonſtratio perbrevis eſt.



[*Quæ ſequuntur ſcripta ſunt in explicationem Epiſtolæ præcedentis.*]

Quod vero attinet ad Inventionem terminorum *p*, *q*, *r*, (*vide* pag. 25 & 8 :) in extractione Radicis affectæ, primum *p* ſic eruo.

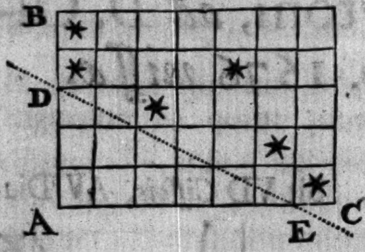
Fig. 1.



x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
0	y	y^2	y^3	y^4

Deſcripto Angulo recto BAC, latera ejus BA, CA divido in partes æquales; & inde normales erigo diſtribuentes angulare ſpatium in æqualia parallelogramma vel quadrata, quæ concipio denominata eſſe a dimenſionibus duarum indefinitarum ſpecierum, puta *x* & *y*, regulariter aſcendentium a termino A; prout vides in Fig. 1. inſcriptas. Ubi *y* denotat Radicem extrahendam; et *x* alteram indefinitam quantitatem, ex cujus poteſtatibus ſeries conficienda eſt. Deinde, cum Æquatio aliqua proponitur, parallelogramma ſingulis ejus terminis correſpondentia inſignio nota aliqua: Et Regula ad duo vel forte plura ex inſignitis parallelogrammis applicata; quorum unum ſit humillimum in columna ſiniſtra juxta AB, & alia ad Regulam dextrorſum ſita, cæteraque omnia non contingentia Regulam ſupra eam jaceant; Seligo terminos Æquationis per parallelogramma contingentia Regulam deſignatos, & inde quæro quantitatem Quotienti addendam.

Fig. 2.



Sic ad extrahendam Radicem y, ex $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$; parallelogramma hujus terminis reſpondentia ſigno nota aliqua $\ast$; ut vides in Fig. 2. Dein applico Regulam DE ad inferiorem e locis ſignatis in ſiniſtra columna; eamque ab inferioribus ad ſuperiora dextrorſum gyrare facio, donec alium ſimiliter vel forte plura e reliquis ſignatis locis cœperit attingere. Videoque loca ſic attacta eſſe x^3, x^2y^2, & y^6. E terminis itaque, $y^6 - 7a^2x^2y^2 + 6a^3x^3$ tanquam nihilo æqualibus (& inſuper ſi placet reductis ad $v^6 - 7v^2 + 6 = 0$, ponendo $y = v\sqrt{ax}$,) quæro valorem y, & invenio quadruplicem, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, & $-\sqrt{2ax}$, quorum quemlibet pro primo termino Quotientis accipere licet, prout e radicibus quampiam extrahere decretum eſt.

Sic Æquatio $y^3 + axy + aay - x^3 - 2a^3 = 0$, quam reſolvebam in priori Epiſtola, dat $-2a^3 + aay + y^3 = 0$, & inde $y = a$ proxime: Cum itaque a ſit primus terminus valoris y, pono p pro cæteris omnibus in infinitum, & ſubſtituo $a + p = y$. (Obvenient hic aliquando difficultates nonnullæ; ſed ex iis, credo, lector ſe proprio marte extricabit.) Subſequentes vero termini q, r, s, &c. eodem modo ex æquationibus ſecundis, tertiis, cæteriſque eruuntur, quo primus p e prima, ſed cura leviori; quia cæteri valores y ſolent prodire dividendo terminum involventem infimam poteſtatem indefinitæ quantitatis x per Coefficientem radicis p, q, r, aut s.

Intellexti credo ex ſuperioribus, regreſſionem ab Areis curvarum ad Lineas rectas, fieri per hanc extractionem Radicis affectæ. Sed duo alii ſunt modi quibus idem perficio.

Eorum unus affinis eſt Computationibus quibus colligebam approximationes ſub finem alterius Epiſtolæ, & intelligi poteſt per hoc exemplum. Proponatur Æquatio ad Aream Hyperbolæ $z = x + \frac{1}{2}xx + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5$, &c. Et partibus ejus multiplicatis in ſe, emerget $z^2 = x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5$, &c. $z^3 = x^3 + \frac{3}{2}x^4 + \frac{7}{4}x^5$, &c. $z^4 = x^4 + 2x^5$, &c. $z^5 = x^5$, &c. Jam de z aufero $\frac{1}{2}zz$, & reſtat $z - \frac{1}{2}zz = x - \frac{1}{6}x^3 - \frac{5}{24}x^4 - \frac{13}{60}x^5$, &c. Huic addo $\frac{1}{6}z^3$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 = x + \frac{1}{24}x^4 + \frac{3}{40}x^5$, &c. Aufero $\frac{1}{24}z^4$, & reſtat $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 = x - \frac{1}{120}x^5$, &c. Addo $\frac{1}{120}z^5$, & fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5 = x$ quamproxime; ſive $x = z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c.

Eodem modo, Series de una indefinita quantitate, in aliam transferri poſſunt. Quemadmodum ſi poſito r radio circuli, x ſinu recto arcus z, & $x + \frac{x^3}{6rr} + \frac{3x^5}{40r^4} +$ &c. longitudine arcus iſtius; atque hanc Seriem a Sinu recto ad Tangentem vellem transferre: Quæro longitudinem Tangentis $\frac{rx}{\sqrt{rr-xx}}$, & reduco in infinitam Seriem $x + \frac{x^3}{2rr} + \frac{3x^5}{8r^4} +$ &c. Vocetur hæc quantitas, t. Colligo poteſtates ejus $t^3 = x^3 + \frac{3x^5}{2rr}$ &c. $t^5 = x^5 +$ &c. Aufero autem t de z, & reſtat $z - t = -\frac{1}{3}x^3 - \frac{3}{10}x^5 -$ &c. Addo $\frac{1}{3}t^3$, & fit $z - t + \frac{1}{3}t^3 = \frac{1}{5}x^5 +$ &c. Aufero $\frac{1}{5}t^5$, & reſtat $z - t + \frac{1}{3}t^3 - \frac{1}{5}t^5 = 0$ quamproxime. Quare eſt $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 -$ &c. Sed ſiquis in uſus Trigonometricos me juſſiſſet exhibere expreſſionem Arcus per Tangentem; eam non hoc circuitu, ſed directa methodo quæſiviſſem.

Per hoc genus Computi colliguntur etiam Series ex duabus vel pluribus indefinitis quantitatibus conſtantes; & Radices affectarum Æquationum magna ex parte extrahuntur. Sed ad hunc poſteriorem uſum adhibeo potius methodum in altera Epiſtola deſcriptam tanquam generaliorem, & (Regulis pro Eliſione ſuperfluorum terminorum habitis) paulo magis expeditam.

Pro Regreſſione vero ab Areis ad Lineas rectas, & ſimilibus, poſſunt hujuſmodi *Theoremata* adhiberi.

THEOREMA I. Sit $z = ay + by^2 + cy^3 + dy^4 + ey^5$, &c.

Et viciſſim erit

$$y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2b^2-ac}{a^5}z^3 + \frac{5abc-5b^3-a^2d}{a^7}z^4 + \frac{3a^2c^2-21ab^2c+6a^2bd+14b^4-a^3e}{a^9}z^5 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Aream Hyperbolæ, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c. Et ſubſtitutis in Regula 1 pro a, $-\frac{1}{2}$ pro b, $\frac{1}{3}$ pro c, $-\frac{1}{4}$ pro d, & $\frac{1}{5}$ pro e; viciſſim exurgit, $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 +$ &c.

THEOREMA II. Sit $z = ay + by^3 + cy^5 + dy^7 + ey^9 +$ &c.

Et viciſſim erit

$$y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3b^2-ac}{a^7}z^5 + \frac{8abc-a^2d-12b^3}{a^{10}}z^7 + \frac{55b^4-55ab^2c+10a^2bd+5a^2c^2-a^3e}{a^{13}}z^9 + \text{\&c.}$$

Exempli gratia. Proponatur Æquatio ad Arcum circuli, $z = y + \frac{y^3}{6rr} + \frac{3y^5}{40r^4} + \frac{5y^7}{112r^6} +$ &c. Et ſubſtitutis in Regula 1 pro a, $\frac{1}{6r^2}$ pro b, $\frac{3}{40r^4}$ pro c, $\frac{5}{112r^6}$ pro d &c; orietur $y = z - \frac{z^3}{6r^2} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} +$ &c.

Fragmentum ** *Epiſtolæ* D. Newtoni, *ad* D. Walliſium *Anno* 1692, *miſſæ.*

SUB finem Epiſtolæ anni 1676 [*Hæc ſunt verba* Walliſii] ſcribit [D. *Newtonus*] etiam Problema determinandi Curvas per conditiones Tangentium in ſua poteſtate eſſe, una cum aliis difficilioribus; ad quæ ſolvenda ſe uſum eſſe dicit duplici methodo, una concinniore, altera generaliore; & utramque literis tranſpoſitis celat: quæ in ordinem redactæ hanc ſententiam exhibent. *Una methodus conſiſtit in Extractione fluentis quantitatis ex æquatione ſimul involvente fluxionem ejus. Altera tantum in aſſumptione ſeriei pro quantitate qualibet incognita ex qua cætera commode derivari poſſunt; & in collatione terminorum homologorum æquationis reſultantis ad eruendos terminos aſſumptæ Seriei.* Harum methodorum Secunda ex verbis jam recitatis abſque ulteriore explicatione intelligi poteſt; priorem ab Authore jam accepi ut ſequitur.

Hæc methodus, ait, ejuſdem eſt generis cum ea pro extrahendo radices ex æquationibus affectis ſuperius deſcripta. Pone quod Problema reſolvendum reducatur ad æquationem fluentes quantitates y & z una cum earum fluxionibus $\dot{y}$ & $\dot{z}$ involventem, & quod fluxio ipſius z uniformis ſit. Ut hæc fluxio ex æquatione evaneſcat, pro ea ponatur unitas, & manebit æquatio ſolas y, z & $\dot{y}$ involvens, *quam Reſolvendam* vocat. Proponitur, inventio ipſius y in Serie infinita convergente, quæ ſolam z involvet. Hoc in aliquibus æquationibus impoſſibile eſt, in aliis præparationem æquationum requirit, ubi vero directe confici poſſit reſolutio eſt hujuſmodi.

PROBLEMA

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* *Extat Epiſtola in Tom. 2. Operum* Walliſii.

PROBLEMA.

Ex æquatione fluxionem radicis involvente radicem extrahere.

RESOLUTIO.

TErmini omnes, ex eodem æquationis latere conſiſtentes, æquentur nihilo, & ipſarum y & y dignitates (ſi opus ſit) exaltentur vel deprimantur, ſic ut earum indices nec alicubi negativi ſint, nec tamen altiores quam ad hunc effectum requiritur; & ſit kz^{λ} terminus infimæ dignitatis eorum qui neque per y neque per ejus fluxionem $\dot{y}$ neque per earum dignitatem quamvis multiplicantur. Sit $lz^{\mu}y^{\alpha}\dot{y}^{\beta}$ terminus alius quilibet, & omnes ordine terminos percurrendo collige ex ſingulis ſeorſim numerum $\frac{\lambda-\mu+\beta}{\alpha+\beta}$ ſic, ut tot habeas ejuſmodi numeros quot ſunt termini. Horum numerorum maximus vocetur ν, & z^{ν} erit dignitas primi termini Seriei. Pro ejus coefficiente ponatur a, & in æquatione quæ *reſolvenda* dicitur ſcribe az^{ν} pro y, & $\nu az^{\nu-1}$ pro $\dot{y}$; ac termini omnes reſultantes in quibus z ejuſdem eſt dignitatis ac in termino kz^{λ}, ſub propriis ſignis collecti, ponantur æquales nihilo. Nam hæc æquatio debite reducta dabit coefficientem a. Sic habes az^{ν} terminum primum Seriei.

OPERATIO SECUNDA.

Pro reliquis omnibus hujus Seriei terminis nondum inventis pone p, & habebis Æquationem $y=az^{\nu}+p$, & inde etiam Æquationem $\dot{y}=\nu az^{\nu-1}+\dot{p}$. In reſolvenda, pro y & $\dot{y}$ ſcribe hos eorum valores & habebis Reſolvendam novam, ubi p officium præſtat ipſius y: & ex hac Reſolvenda primum extrahes terminum Seriei p eodem modo atque terminum primum Seriei totius $y=az^{\nu}+p$ ex Reſolvenda prima extraxiſti.

OPERATIO TERTIA ET SEQUENTES.

Dein tertiam Reſolvendam eadem ratione invenias atque ſecundam inveniſti, & ex ea terminum tertium Seriei totius extrahes. Et ſimiliter Reſolvendam quartam invenies, & ex ea quartum Seriei terminum, & ſic in infinitum. Series autem ſic inventa erit radix Æquationis quam extrahere oportuit.

EXEMPLUM.

EXEMPLUM.

Ex Æquatione $y^2\dot{z}^2 - z^2\dot{z}\dot{y} - d^2\dot{z}\dot{z} + dz\dot{z}^2 = 0$, extrahenda ſit radix y. Pone $\dot{z} = 1$, & Æquatio evadet $y^2 - z^2\dot{y} - dd + dz = 0$, quæ eſt Reſolvenda. Jam vero terminus infimus in quo nec y neque $\dot{y}$ reperitur, eſt dd, qui ipſi kz^λ æquatus dat $\lambda = 0$. Terminis reliquis y^2, $-z^2\dot{y}$ pone $lz^\mu y^\alpha \dot{y}^\beta$ æqualem ſucceſſive, & inde in primo caſu habebis $\mu = 0$, $\alpha = 2$, $\beta = 0$, in ſecundo $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ fit in primo caſu 0, in ſecundo -1. Unde ν eſt 0, & az^ν & $\nu az^{\nu - 1}$ ſunt a & 0; quarum ultimæ duæ a & 0 in Reſolvenda pro y & $\dot{y}$ ſcriptæ, producunt $aa + 0z^2 - dd + dz$; & termini aa & $-dd$, in quibus index dignitatis z eſt λ ſeu 0, poſiti æquales nihilo dant $a = d$. Unde primus Seriei terminus az^ν evadit d.

OPERATIO SECUNDA.

Pro terminis reliquis pone p, & habebis æquationem $y = d + p$, & inde $\dot{y} = \dot{p}$; qui valores in Reſolvenda pro y & $\dot{y}$ ſubſtituti dant Reſolvendam novam $2dp + pp - zz\dot{p} + dz = 0$, ubi p & $\dot{p}$ vices ſubeunt ipſarum y & $\dot{y}$. Terminus unicus in quo nec p neque $\dot{p}$ reperitur eſt dz, qui cum termino kz^λ collatus dat $\lambda = 1$. Terminis reliquis $2dp$, pp & $-zz\dot{p}$ pone $lz^\mu p^\alpha \dot{p}^\beta$ æqualem ſucceſſive; & inde in primo caſu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in ſecundo $\mu = 0$, $\alpha = 2$, & $\beta = 0$; & in tertio $\mu = 2$, $\alpha = 0$, & $\beta = 1$. Et hinc $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit primo caſu 1, in ſecundo $\frac{1}{2}$, in tertio 0. Unde ν eſt 1, & az^ν & $\nu az^{\nu - 1}$ ſunt az & a. Termini duo ultimi az & a in Reſolvenda pro p & $\dot{p}$ reſpective ſcripti, producunt $2daz + a^2z^2 - az^2 + dz$. Et termini $2daz$ & dz in quibus index dignitatis z eſt λ ſeu 1, poſiti æquales nihilo, dant $a = -\frac{1}{2}$. Unde az^ν terminus primus Seriei p fit $-\frac{1}{2}z$.

OPERATIO TERTIA.

Pro terminis reliquis nondum inventis pone q & habebis æquationem $p = -\frac{1}{2}z + q$, & inde $\dot{p} = -\frac{1}{2} + \dot{q}$: Qui valores pro p & $\dot{p}$ in Reſolvenda noviſſima ſubſtituti producunt Reſolvendam novam $2dq - zq + qq + \frac{3}{4}zz - zz\dot{q} = 0$. Ubi q & $\dot{q}$ vices ſupplent ipſorum y & $\dot{y}$. Terminus unicus in quo neque q nec $\dot{q}$ reperitur eſt $\frac{3}{4}zz$, qui cum kz^λ collatus dat $\lambda = 2$. Terminis reliquis $2dq$, $-zq$, $+qq$, $-zz\dot{q}$ pone $lz^\mu q^\alpha \dot{q}^\beta$ æqualem ſucceſſive; & inde in primo caſu habebis $\mu = 0$, $\alpha = 1$, & $\beta = 0$; in ſecundo,

$\mu = 1$,

$\mu = 1$, $\alpha = 1$, $\beta = 0$; in tertio, $\mu = 0$, $\alpha = 2$, $\beta = 0$: in quarto $\mu = 2$, $\alpha = 0$, $\beta = 1$: & inde $\frac{\lambda - \mu + \beta}{\alpha + \beta}$ evadit in primo caſu 2, in ſecundo, tertio, & quarto 1. Et hinc ν eſt 2, vel az^{ν} & $\nu az^{\nu - 1}$ ſunt az^2 & $2az$: qui valores in Reſolvenda pro q & $\dot{q}$ ſubſtituti dant $2daz^2 - az^3 + aaz^4 + \frac{3}{4}zz - 2az^3$; & termini $2dazz + \frac{3}{4}zz$ in quibus index dignitatis z eſt λ ſeu 2, poſiti æquales nihilo, dant $a = -\frac{3}{8d}$. Unde az^{ν} terminus primus Seriei q evadit $-\frac{3zz}{8d}$.

OPERATIO QUARTA.

Pro reliquis Seriei terminis nondum inventis pone r, & habebis æquationes $q = -\frac{3zz}{8d} + r$, & $\dot{q} = -\frac{3z}{4d} + \dot{r}$; & inde reſolvendam novam $2d\dot{r} + \frac{9z^3}{8d} - zr + \frac{9z^4}{64dd} - \frac{3zzr}{4d} + rr - zz\dot{r} = 0$; & ex ea per Methodum ſuperiorem habebis $-\frac{9z^3}{16dd}$ terminum primum Seriei r. Et ſic pergitur in infinitum.

Eſt igitur radix extrahenda $y = d + p = d - \frac{1}{2}z + q = d - \frac{1}{2}z - \frac{3zz}{8d} + r = d - \frac{1}{2}z - \frac{3zz}{8d} - \frac{9z^3}{16dd} -$ &c. Et operationem continuando producere licet radicem ad terminos plures.

Et eadem methodo, dicit *Newtonus*, radices æquationum, fluxiones ſecundas, tertias, quartas, ($\ddot{y}$, $\dddot{y}$, $\ddddot{y}$,) aliaſque involventium, extrahi poſſe.

His utitur radicum extractionibus ubi aliæ Methodi nil proſunt. Nam in Epiſtola prædicta anni 1676 docet, quod in Solutione problematum de Tangentibus inverſorum, caſus aliqui dantur in quibus hæc Methodus generalis non requiritur: & particulariter, ſi in triangulo rectangulo quod ab ordinata, tangente, & interjacente parte abſciſſæ conſtituitur, relatio duorum quorumlibet e lateribus tribus per æquationem quamvis definiatur; Problema abſque Methodo hacce generali ſolvi poterit.

Methodi autem hæ omnes tam particulares quam generales collectim ſumptæ, ſolutionem exhibent ſecundæ partis problematis, quod *Newtonus* ſub initio iſtius Epiſtolæ his verbis propoſuit. *Data æquatione quotcunque fluentes quantitates involvente fluxiones invenire, & vice verſa.* Nam tota fluxionum Methodus in hujus directa & inverſa ſolutione conſiſtit.

Part of a Letter from Sir *Iſ. Newton*, to Mr. *J. Collins*, *Novemb.* 8. 1676.

There is no Curve-line expreſs'd by any Equation of three terms, tho' the unknown quantities affect one another in it, or the Indices of their Dignities be ſurd quantities (ſuppoſe $ax^\lambda + bx^\mu y^\sigma + cy^\tau = 0$, *where* x *ſignifies the Baſe,* y *the Ordinate,* λ, μ, σ, τ *the Indices of the dignities of* x *and* y, *and* a, b, c *known quantities with their ſigns* + *or* —, *) I ſay, there is no ſuch Curve-line, but I can, in leſs than half a quarter of an hour, tell whether it may be Squar'd, or what are the ſimpleſt Figures it may be compar'd with, be thoſe Figures Conic Sections, or others: And then by a direct and ſhort way, (I dare ſay the ſhorteſt the nature of the thing admits of, for a general one,) I can compare them. And ſo if any two Figures expreſs'd by ſuch Equations be propounded, I can, by the ſame Rule compare them if they may be compar'd. This may ſeem a bold aſſertion, becauſe it's hard to ſay a Figure may, or may not, be Squar'd, or Compar'd with another; but it's plain to me by the fountain I draw it from, tho' I will not undertake to prove it to others. The ſame Method extends to Equations of four Terms, and others alſo, but not ſo generally.*

Fragmentum Epiſtolæ D. Newtoni *ad D.* Collinſium, Novemb. 8. 1676, *Latine redditum.*

Nulla extat Curva cujus Æquatio ex tribus conſtat terminis, in qua, licet quantitates incognitæ ſe mutuo afficiant, vel Indices dignitatum ſint ſurdæ quantitates (v. g. $ax^\lambda + bx^\mu y^\sigma + cy^\tau = 0$, ubi *x* deſignat Baſin, *y* Ordinatam, λ, μ, σ, τ Indices dignitatum ipſius *x* & *y*, & *a*, *b*, *c* quantitates cognitas una cum ſignis ſuis + vel —) nulla inquam hujuſmodi eſt Curva, de qua, an Quadrari poſſit, necne, vel quænam ſint Figuræ ſimpliciſſimæ quibuſcum comparari poſſit, ſive ſint Conicæ ſectiones ſive aliæ magis complicatæ, intra horæ octantem reſpondere non poſſim. Deinde * methodo directa & brevi, imo methodorum omnium generalium breviſſima eas comparare queo. Quinetiam ſi duæ quævis Figuræ per hujuſmodi Æquationes expreſſæ proponantur, per eandem Regulam, eas, modo comparari poſſint, comparo.

Affirmatio quidem videri poteſt temeraria, propterea quod perdifficile ſit dictu an Figura Quadrari vel cum alia comparari poſſit, necne; mihi autem manifeſtum eſt, ex eo unde deduxi fonte, quanquam id aliis demonſtrare in me ſuſcipere nollem. Eadem methodus Æquationes quatuor terminorum aliaſque complectitur, haud tamen adeo generaliter.

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* *Methodum habes in Coroll.* 2. *Prop.* 10. *Tract. ſequentis.*

TRACTATUS
DE
Quadratura Curvarum.

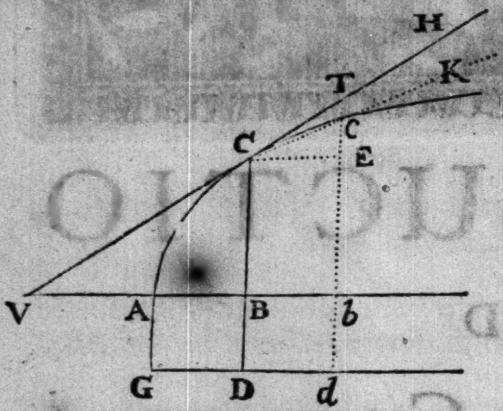
INTRODUCTIO
AD
Quadraturam Curvarum.

Quantitates Mathematicas non ut ex partibus quam minimis conſtantes, ſed ut motu continuo deſcriptas hic conſidero. Lineæ deſcribuntur ac deſcribendo generantur non per appoſitionem partium ſed per motum continuum punctorum, ſuperficies per motum linearum, ſolida per motum ſuperficierum, anguli per rotationem laterum, tempora per fluxum continuum, & ſic in cæteris. Hæ Geneſes in rerum natura locum vere habent & in motu corporum quotidie cernuntur. Et ad hunc modum Veteres ducendo rectas mobiles in longitudinem rectarum immobilium geneſin docuerunt rectangulorum.

Conſiderando igitur quod quantitates æqualibus temporibus creſcentes & creſcendo genitæ, pro velocitate majori vel minori qua creſcunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; & has motuum vel incrementorum velocitates nominando *Fluxiones* & quantitates genitas nominando *Fluentes*, incidi paulatim *Annis* 1665 & 1666 in Methodum Fluxionum qua hic uſus ſum in Quadratura Curvarum.

Fluxiones ſunt quam proxime ut Fluentium augmenta æqualibus temporis particulis quam minimis genita, &, ut accurate loquar, ſunt in prima ratione augmentorum naſcentium; exponi autem poſſunt per lineas quaſcunque quæ ſunt ipſis proportionales.

Ut ſi areæ ABC, ABDG Ordinatis BC, BD ſuper baſi AB uniformi cum motu progredientibus deſcribantur, harum arearum fluxiones erunt inter ſe ut Ordinatæ deſcribentes BC & BD, & per Ordinatas illas exponi poſſunt, propterea quod Ordinatæ illæ ſunt ut arearum augmenta naſcentia.



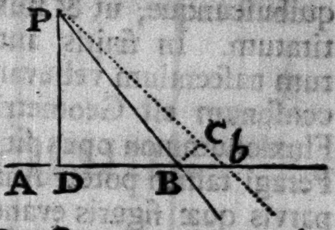
Progrediatur ordinata BC de loco ſuo BC in locum quemvis novum bc. Compleatur parallelogrammum BCEb, ac ducatur recta VTH quæ Curvam tangat in C ipſiſque bc & BA productis occurrat in T & V: & Abſciſſæ AB, Ordinatæ BC, & Lineæ Curvæ ACc augmenta modo genita erunt Bb, Ec & Cc; & in horum augmentorum naſcentium ratione prima ſunt latera trianguli CET, ideoque fluxiones ipſarum AB, BC & AC ſunt ut trianguli illius CET latera CE, ET & CT & per eadem latera exponi poſſunt, vel quod perinde eſt per latera trianguli conſimilis VBC.

Eodem recidit ſi ſumantur fluxiones in ultima ratione partium evaneſcentium. Agatur recta Cc & producatur eadem ad K. Redeat Ordinata bc in locum ſuum priorem BC, & coeuntibus punctis C & c, recta CK coincidet cum tangente CH, & triangulum evaneſcens CEc in ultima ſua forma evadet ſimile triangulo CET, & ejus latera evaneſcentia CE, Ec & Cc erunt ultimo inter ſe ut ſunt trianguli alterius CET latera CE, ET & CT, & propterea in hac ratione ſunt fluxiones linearum AB, BC & AC. Si puncta C & c parvo quovis intervallo ab invicem diſtant recta CK parvo intervallo a tangente CH diſtabit. Ut recta CK cum tangente CH coincidat & rationes ultimæ linearum CE, Ec & Cc inveniantur, debent puncta C & c coire & omnino coincidere. Errores quam minimi in rebus Mathematicis non ſunt contemnendi.

Simili argumento ſi circulus centro B radio BC deſcriptus in longitudinem Abſciſſæ AB ad angulos rectos uniformi cum motu ducatur, fluxio ſolidi geniti ABC erit ut circulus ille generans, & fluxio ſuperficiei ejus erit ut perimeter Circuli illius & fluxio lineæ curvæ AC conjunctim.

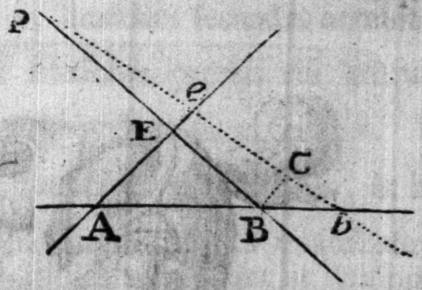
Nam quo tempore ſolidum ABC generatur ducendo circulum illum in longitudinem abſciſſæ AB, eodem ſuperficies ejus generatur ducendo perimetrum circuli illius in longitudinem Curvæ AC. Hujus methodi accipe etiam exempla quæ ſequuntur.

Recta PB circa polum datum P revolvens ſecet aliam poſitione datam rectam AB: quæritur proportio fluxionum rectarum illarum AB & PB.



Progrediatur recta PB de loco ſuo PB in locum novum Pb. In Pb capiatur PC ipſi PB æqualis, & ad AB ducatur PD ſic, ut angulus bPD æqualis ſit angulo bBC; & ob ſimilitudinem triangulorum bBC, bPD erit augmentum Bb ad augmentum Cb ut Pb ad Db. Redeat jam Pb in locum ſuum priorem PB ut augmenta illa evaneſcant, & evaneſcentium ratio ultima, id eſt ratio ultima Pb ad Db, ea erit quæ eſt PB ad DB, exiſtente angulo PDB recto, & propterea in hac ratione eſt fluxio ipſius AB ad fluxionem ipſius PB.

Recta PB circa datum Polum P revolvens ſecet alias duas poſitione datas rectas AB & AE in B & E: quæritur proportio fluxionum rectarum illarum AB & AE.



Progrediatur recta revolvens PB de loco ſuo PB in locum novum Pb rectas AB, AE in punctis b & e ſecantem, & rectæ AE parallela BC ducatur ipſi Pb occurrens in C, & erit Bb ad BC ut Ab ad Ae, & BC ad Ee ut PB ad PE, & conjunctis rationibus Bb ad Ee ut Ab×PB ad Ae×PE. Redeat jam linea Pb in locum ſuum priorem PB, & augmentum evaneſcens Bb erit ad augmentum evaneſcens Ee ut AB×PB ad AE×PE, ideoque in hac ratione eſt fluxio rectæ AB ad fluxionem rectæ AE.

Hinc ſi recta revolvens PB lineas quaſvis Curvas poſitione datas ſecet in punctis B & E, & rectæ jam mobiles AB, AE Curvas illas tangant in Sectionum punctis B & E: erit fluxio Curvæ quam recta AB tangit ad fluxionem Curvæ quam recta AE tangit ut AB×PB ad AE×PE. Id quod etiam eveniet ſi recta PB Curvam aliquam poſitione datam perpetuo tangat in puncto mobili P.

Fluat quantitas x *uniformiter & invenienda ſit fluxio quantitatis* x^n. Quo tempore quantitas x fluendo evadit $x+o$, quantitas x^n evadet $\overline{x+o}|^n$, id eſt per methodum ſerierum infinitarum, $x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} + \&c.$

Et augmenta o & $nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +$ &c. ſunt ad invicem ut 1 & $nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +$ &c. Evaneſcant jam augmenta illa, & eorum ratio ultima erit 1 ad nx^{n-1} : ideoque fluxio quantitatis x eſt ad fluxionem quantitatis x^n ut 1 ad nx^{n-1}.

Similibus argumentis per methodum rationum primarum & ultimarum colligi poſſunt fluxiones linearum ſeu rectarum ſeu curvarum in caſibus quibuſcunque, ut & fluxiones ſuperficierum, angulorum & aliarum quantitatum. In finitis autem quantitatibus Analyſin ſic inſtituere, & finitarum naſcentium vel evaneſcentium rationes primas vel ultimas inveſtigare, conſonum eſt Geometriæ Veterum : & volui oſtendere quod in Methodo Fluxionum non opus ſit figuras infinite parvas in Geometriam introducere. Peragi tamen poteſt Analyſis in figuris quibuſcunque ſeu finitis ſeu infinite parvis quæ figuris evaneſcentibus finguntur ſimiles, ut & in figuris quæ per Methodos Indiviſibilium pro infinite parvis haberi ſolent, modo caute procedas.

Ex Fluxionibus invenire Fluentes Problema difficilius eſt, & ſolutionis primus gradus æquipollet Quadraturæ Curvarum ; de qua ſequentia olim ſcripſi.

DE

Quadratura Curvarum.

Quantitates indeterminatas ut motu perpetuo creſcentes vel decreſcentes, id eſt ut fluentes vel defluentes in ſequentibus conſidero, deſignoque literis z, y, x, v, & earum fluxiones ſeu celeritates creſcendi noto iiſdem literis punctatis $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$, Sunt & harum fluxionum fluxiones ſeu mutationes magis aut minus celeres quas ipſarum z, y, x, v, fluxiones ſecundas nominare licet & ſic deſignare $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, & harum fluxiones primas ſeu ipſarum z, y, x, v fluxiones tertias ſic $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$, & quartas ſic $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$. Et quemadmodum $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ ſunt fluxiones quantitatum $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, & hæ ſunt fluxiones quantitatum $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$, & hæ ſunt fluxiones quantitatum primarum z, y, x, v : ſic hæ quantitates conſiderari poſſunt ut fluxiones aliarum quas ſic deſignabo, $\overset{\prime}{z}$, $\overset{\prime}{y}$, $\overset{\prime}{x}$, $\overset{\prime}{v}$, & hæ ut fluxiones aliarum $\overset{\prime\prime}{z}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{x}$, $\overset{\prime\prime}{y}$, & hæ ut fluxiones aliarum $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{v}$. Deſignant igitur $\overset{\prime\prime}{z}$, $\overset{\prime}{z}$, z, $\dot{z}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, $\overset{\cdot\cdot\cdot\cdot\cdot}{z}$, &c. ſeriem quantitatum quarum quælibet poſterior eſt fluxio præcedentis & quælibet prior eſt fluens quantitas fluxionem habens ſubſequentem. Similis eſt ſeries $\sqrt{\overset{\prime\prime}{az-zz}}$, $\sqrt{\overset{\prime}{az-zz}}$, $\sqrt{az-zz}$, $\sqrt{\overset{\cdot}{az-zz}}$, $\sqrt{\overset{\cdot\cdot}{az-zz}}$, $\sqrt{\overset{\cdot\cdot\cdot}{az-zz}}$, ut & ſeries $\frac{az+zz}{\overset{\prime\prime}{a-z}}$, $\frac{az+zz}{\overset{\prime}{a-z}}$, $\frac{az+zz}{a-z}$, $\frac{az+zz}{\overset{\cdot}{a-z}}$, $\frac{az+zz}{\overset{\cdot\cdot}{a-z}}$, $\frac{az+zz}{\overset{\cdot\cdot\cdot}{a-z}}$.

Et notandum eſt quod quantitas quælibet prior in his ſeriebus eſt ut area figuræ curviliniæ cujus ordinatim applicata rectangula eſt quantitas poſterior & abſciſſa eſt z: uti $\sqrt{\overset{\prime}{az-zz}}$ area curvæ cujus ordinata eſt $\sqrt{az-zz}$ & abſciſſa z. Quo autem ſpectant hæc omnia patebit in Propoſitionibus quæ ſequuntur.

PROP. I. PROB. I.

Data æquatione quotcunque ſluentes quantitates involvente, invenire ſluxiones.

Solutio.

Multiplicetur omnis æquationis terminus per indicem dignitatis quantitatis cujuſque fluentis quam involvit, & in ſingulis multiplicationibus mutetur dignitatis latus in fluxionem ſuam, & aggregatum factorum omnium ſub propriis ſignis erit æquatio nova.

Explicatio.

Sunto a, b, c, d &c. quantitates determinatæ & immutabiles, & proponatur æquatio quævis quantitates fluentes z, y, x &c. involvens, uti $x^3 - xy^2 + a^2z - b^3 = 0$. Multiplicentur termini primo per indices dignitatum x, & in ſingulis multiplicationibus pro dignitatis latere, ſeu x unius dimenſionis, ſcribatur $\dot{x}$, & ſumma factorum erit $3\dot{x}x^2 - \dot{x}y^2$. Idem fiat in y & prodibit $-2x\dot{y}y$. Idem fiat in z & prodibit $aa\dot{z}$. Ponatur ſumma factorum æqualis nihilo, & habebitur æquatio $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Dico quod hac æquatione definitur relatio fluxionum.

Demonſtratio.

Nam ſit o quantitas admodum parva & ſunto $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, quantitatum z, y, x momenta id eſt incrementa momentanea ſynchrona. Et ſi quantitates fluentes jam ſunt z, y & x hæ poſt momentum temporis incrementis ſuis $o\dot{z}$, $o\dot{y}$, $o\dot{x}$ auctæ, evadent $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$, quæ in æquatione prima pro z, y, & x ſcriptæ dant æquationem $x^3 + 3x^2o\dot{x} + 3xo^2\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2z + a^2o\dot{z} - b^3 = 0$.

Subducatur æquatio prior, & reſiduum diviſum per o erit $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$. Minuatur quantitas o in infinitum, & neglectis terminis evaneſcentibus reſtabit $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Q. E. D.

Explicatio plenior.

Ad eundem modum ſi æquatio eſſet $x^3 - xy^2 + a^2\sqrt{ax-yy} - b^3 = 0$, produceretur $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{\overline{\sqrt{ax-yy}}} = 0$. Ubi ſi fluxionem $\dot{\overline{\sqrt{ax-yy}}}$ tollere velis, pone $\sqrt{ax-yy} = z$, & erit $ax - y^2 = z^2$, & (per hanc Propoſitionem) $a\dot{x} - 2\dot{y}y = 2\dot{z}z$ ſeu $\frac{a\dot{x}-2\dot{y}y}{2z} = \dot{z}$, hoc eſt $\frac{a\dot{x}-2\dot{y}y}{2\sqrt{ax-yy}} = \dot{\overline{\sqrt{ax-yy}}}$. Et inde $3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x}-2a^2\dot{y}y}{2\sqrt{ax-yy}} = 0$.

Et per operationem repetitam pergitur ad fluxiones ſecundas, tertias & ſequentes. Sit æquatio $zy^3 - z^4 + a^4 = 0$, & fiet per operationem primam $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, per ſecundam $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = 0$, per tertiam $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$.

Ubi vero ſic pergitur ad fluxiones ſecundas, tertias & ſequentes, convenit quantitatem aliquam ut uniformiter fluentem conſiderare, & pro ejus fluxione prima unitatem ſcribere, pro ſecunda vero & ſequentibus nihil. Sit æquatio $zy^3 - z^4 + a^4 = 0$, ut ſupra; & fluat z uniformiter, ſitque ejus fluxio unitas, & fiet per operationem primam $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$, per ſecundam $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 12z^2 = 0$, per tertiam $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = 0$.

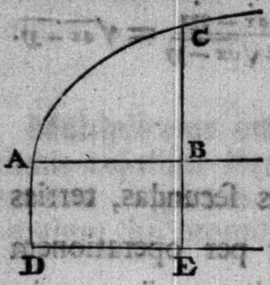
In hujus autem generis æquationibus concipiendum eſt quod fluxiones in ſingulis terminis ſint ejuſdem ordinis, id eſt vel omnes primi ordinis $\dot{y}$, $\dot{z}$, vel omnes ſecundi $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$, vel omnes tertii $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. Et ubi res aliter ſe habet complendus eſt ordo per ſubintellectas fluxiones quantitatis uniformiter fluentis. Sic æquatio noviſſima complendo ordinem tertium fit $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24\dot{z}z^3 = 0$.

PROP.

PROP. II. PROB. II.

Invenire Curvas quæ quadrari poſſunt.

Sit ABC figura invenienda, BC Ordinatim applicata rectangula, & AB abſciſſa. Producatur CB ad E ut ſit BE = 1, & compleatur parallelogrammum ABED: & arearum ABC, ABED fluxiones erunt ut BC & BE. Aſſumatur igitur æquatio quævis qua relatio arearum definiatur, & inde dabitur relatio ordinatarum BC & BE per Prop. 1. Q. E. I.



Hujus rei exempla habentur in Propoſitionibus duabus ſequentibus.

PROP. III. THEOR. I.

Si pro abſciſſa AB & area AE ſeu AB × 1 promiſcue ſcribatur z, & ſi pro $e + fz^n + gz^{2n} + hz^{3n} +$ &c. ſcribatur R: ſit autem area Curvæ $z^\theta R^\lambda$, erit ordinatim applicata BC æqualis

$$\overline{\theta e + \begin{matrix}\theta\\ +\lambda n\end{matrix} \times fz^n + \begin{matrix}\theta\\ +2\lambda n\end{matrix} \times gz^{2n} + \begin{matrix}\theta\\ +3\lambda n\end{matrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1}.$$

Demonſtratio.

Nam ſi ſit $z^\theta R^\lambda = v$, erit (per Prop. 1) $\theta\dot{z}z^{\theta-1}R^\lambda + \lambda z^\theta \dot{R} R^{\lambda-1} = \dot{v}$. Pro R^λ in primo æquationis termino & z^θ in ſecundo ſcribe $RR^{\lambda-1}$ & $zz^{\theta-1}$, & fiet $\theta\dot{z}R + \lambda z\dot{R}$ in $z^{\theta-1}R^{\lambda-1} = \dot{v}$. Erat autem $R = e + fz^n + gz^{2n} + hz^{3n} +$ &c. et inde (per Prop. 1) fit $\dot{R} = nf\dot{z}z^{n-1} + 2ng\dot{z}z^{2n-1} + 3nh\dot{z}z^{3n-1} +$ &c. quibus ſubſtitutis & ſcripta BE ſeu 1 pro $\dot{z}$, fiet

$$\overline{\theta e + \begin{matrix}\theta\\ +\lambda n\end{matrix} \times fz^n + \begin{matrix}\theta\\ +2\lambda n\end{matrix} \times gz^{2n} + \begin{matrix}\theta\\ +3\lambda n\end{matrix} \times hz^{3n} + \&c.} \times z^{\theta-1}R^{\lambda-1} = \dot{v} = BC.$$

Q. E. D.

PROP.

PROP. IV. THEOR. II.

Si Curvæ abſciſſa AB ſit z, & ſi pro $e+fz^{n}+gz^{2n}+$ &c. ſcribatur R, & pro $k+lz^{n}+mz^{2n}+$ &c. ſcribatur S; ſit autem area Curvæ $z^{\theta}R^{\lambda}S^{\mu}$: Erit ordinatim applicata BC æqualis

$$\left.\begin{array}{lllll}
\theta ek & +\begin{array}{l}\theta\\ +\lambda n\end{array}\times fkz^{n} & +\begin{array}{l}\theta\\ +2\lambda n\end{array}\times gkz^{2n} & \ldots * \ldots & \ldots * \ldots\\
 & +\begin{array}{l}\theta\\ +\mu n\end{array}\times elz^{n} & +\begin{array}{l}\theta\\ +\lambda n\\ +\mu n\end{array}\times flz^{2n} & +\begin{array}{l}\theta\\ +2\lambda n\\ +\mu n\end{array}\times glz^{3n} & \ldots * \ldots\\
 & & +\begin{array}{l}\theta\\ +2\mu n\end{array}\times emz^{2n} & +\begin{array}{l}\theta\\ +\lambda n\\ +2\mu n\end{array}\times fmz^{3n} & +\begin{array}{l}\theta\\ +2\lambda n\\ +2\mu n\end{array}\times gmz^{4n}
\end{array}\right\}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$$

Demonſtratur ad modum Propoſitionis ſuperioris.

PROP. V. THEOR. III.

Si Curvæ abſciſſa AB ſit z, & pro $e+fz^{n}+\overline{gz^{2n}+hz^{3n}+\text{\&c.}}$ ſcribatur R: ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^{n}+cz^{2n}+dz^{3n}+\text{\&c.}}$ & ponatur $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=v$, &c.

$$\begin{array}{ll}
\text{Erit Area}=z^{\theta}R^{\lambda}\text{ in} & +\dfrac{\frac{1}{n}a}{re}\\[2ex]
 & +\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}z^{n}\\[2ex]
 & +\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}z^{2n}\\[2ex]
 & +\dfrac{\frac{1}{n}d-\overline{s+2}\times fC-\overline{t+1}\times gB-vhA}{\overline{r+3}\times e}z^{3n}\\[2ex]
 & +\dfrac{-\overline{s+3}\times fD-\overline{t+2}\times gC-\overline{v+1}\times hB}{\overline{r+4}\times e}z^{4n}\\[2ex]
 & +\text{\&c.}
\end{array}$$

Ubi A, B, C, D, &c. denotant totas coefficientes datas terminorum ſingulorum in ſerie cum ſignis ſuis + & —, nempe

A primi termini coefficientem $\dfrac{\frac{1}{n}a}{re}$

B ſecundi termini coefficientem $\dfrac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}$

C tertii termini coefficientem $\dfrac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}$

Et ſic deinceps.

Demonſtratio.

Sunto juxta Propoſitionem tertiam.

CURVARUM ORDINATÆ AREÆ

$$\left.\begin{array}{llllll}
1\ . & \theta eA & +\ \theta \times fAz^{n} & +\ \theta \times gAz^{2n} & +\ \theta \times hAz^{3n}, & \&c. \\
 & & +\lambda n & +2\lambda n & +3\lambda n & \\
2\ .\ .\ . & & +\ \overline{\theta+n} \times eBz^{n} & +\ \overline{\theta+n} \times fBz^{2n} & +\ \overline{\theta+n} \times gBz^{3n} & \&c. \\
 & & & +\ \lambda n & +\ 2\lambda n & \\
3\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . & & & +\ \overline{\theta+2n} \times eCz^{2n} & +\ \overline{\theta+2n} \times fCz^{3n} & \&c. \\
 & & & & +\ \lambda n & \\
4\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . & & & & +\ \overline{\theta+3n} \times eDz^{3n} & \&c.
\end{array}\right\} \times z^{\theta-1}R^{\lambda-1}.\ \left\{\begin{array}{l}
Az^{\theta}R^{\lambda} \\ \\
Bz^{\theta+n}R^{\lambda} \\ \\
Cz^{\theta+2n}R^{\lambda} \\ \\
Dz^{\theta+3n}R^{\lambda}
\end{array}\right.$$

Et ſi ſumma Ordinatarum ponatur æqualis Ordinatæ $a + bz^{n} + cz^{2n} + dz^{3n} + \&c.$ in $z^{\theta-1}R^{\lambda-1}$, ſumma arearum $z^{\theta}R^{\lambda}$ in $A + Bz^{n} + Cz^{2n} + Dz^{3n} + \&c.$ æqualis erit areæ Curvæ cujus iſta eſt Ordinata. Æquentur igitur Ordinatarum termini correſpondentes,

et fiet $a = \theta eA$,

$$b = \overline{\theta+\lambda n} \times fA + \overline{\theta+n} \times eB,$$

$$c = \overline{\theta+2\lambda n} \times gA + \overline{\theta+n+\lambda n} \times fB + \overline{\theta+2n} \times eC,$$

&c.

et inde $A = \frac{a}{\theta e}$

$$B = \frac{b - \overline{\theta+\lambda n} \times fA}{\overline{\theta+n} \times e}$$

$$C = \frac{c - \overline{\theta+2\lambda n} \times gA - \overline{\theta+n+\lambda n} \times fB}{\overline{\theta+2n} \times e}$$

Et ſic deinceps in infinitum.

Pone jam $\frac{\theta}{n} = r$, $r + \lambda = s$, $s + \lambda = t$, &c. et in area $z^{\theta}R^{\lambda}$ in $A + Bz^{n} + Cz^{2n} + Dz^{3n} + \&c.$ ſcribe ipſorum A, B, C, &c. valores inventos, & prodibit ſeries propoſita. Q. E. D.

Et notandum eſt quod Ordinata omnis duobus modis in ſeriem reſolvitur. Nam index n vel affirmativus eſſe poteſt vel negativus. Proponatur Ordinata $\frac{3k - lzz}{zz\sqrt{kz - lz^{3} + mz^{4}}}$: Hæc vel ſic ſcribi poteſt

$$z^{-\frac{5}{2}} \times \overline{3k - lz^{2}} \times \overline{k - lz^{2} + mz^{3}}\big|^{-\frac{1}{2}},$$

vel ſic $z^{-2} \times \overline{-l + 3kz^{-2}} \times \overline{m - lz^{-1} + kz^{-3}}\big|^{-\frac{1}{2}}$.

In caſu priore eſt $a = 3k$, $b = 0$, $c = -l$, $e = k$, $f = 0$, $g = -l$, $h = m$, $\lambda = \frac{1}{2}$, $n = 1$, $\theta - 1 = -\frac{5}{2}$, $\theta = -\frac{3}{2} = r$, $s = -1$, $t = -\frac{1}{2}$, $v = 0$.

In poſteriore eſt $a = -l$, $b = 0$, $c = 3k$, $e = m$, $f = -l$, $g = 0$, $h = k$, $\lambda = \frac{1}{2}$, $n = -1$, $\theta - 1 = -2$, $\theta = -1$, $r = 1$, $s = 1\frac{1}{2}$, $t = 2$, $v = 2\frac{1}{2}$. Tentandus eſt caſus uterque. Et ſi ſerierum alterutra ob terminos tandem deficientes abrumpitur ac terminatur habebitur area Curvæ in terminis finitis. Sic in exempli hujus priore caſu ſcribendo in ſerie valores ipſorum a, b, c, e, f, g, h, λ, θ, r, s, t, v, termini omnes poſt primum evaneſcunt in infinitum & area Curvæ prodit $-2\sqrt{\frac{k - lz^2 + mz^3}{z^3}}$. Et hæc area ob ſignum negativum adjacet abſciſſæ ultra ordinatam productæ. Nam area omnis affirmativa adjacet tam abſciſſæ quam ordinatæ, negativa vero cadit ad contrarias partes ordinatæ & adjacet abſciſſæ productæ, manente ſcilicet ſigno Ordinatæ. Hoc modo ſeries alterutra & nonnunquam utraque ſemper terminatur & finita evadit ſi Curva geometrice quadrari poteſt. At ſi Curva talem quadraturam non admittit, ſeries utraque continuabitur in infinitum, & earum altera converget & aream dabit approximando, præterquam ubi r (propter aream infinitam) vel nihil eſt vel numerus integer & negativus, vel ubi $\frac{z^n}{e}$ æqualis eſt unitati. Si $\frac{z^n}{e}$ minor eſt unitate, converget ſeries in qua index n affirmativus eſt: ſin $\frac{z^n}{e}$ unitate major eſt, converget ſeries altera. Si in uno caſu area adjacet abſciſſæ ad uſque ordinatam ductæ, in altero adjacet abſciſſæ ultra ordinatam productæ.

Nota inſuper quod ſi Ordinata contentum eſt ſub factore rationali $\mathcal{Q}$ & factore ſurdo irreducibili R^π & factoris ſurdi latus R non dividit factorem rationalem $\mathcal{Q}$; erit $\lambda - 1 = \pi$ & $R^{\lambda - 1} = R^\pi$. Sin factoris ſurdi latus R dividit factorem rationalem ſemel, erit $\lambda - 1 = \pi + 1$ & $R^{\lambda - 1} = R^{\pi + 1}$: ſi dividit bis, erit $\lambda - 1 = \pi + 2$ & $R^{\lambda - 1} = R^{\pi + 2}$: ſi ter, erit $\lambda - 1 = \pi + 3$, & $R^{\lambda - 1} = R^{\pi + 3}$: & ſic deinceps.

Si Ordinata eſt fractio rationalis irreducibilis cum denominatore ex duobus vel pluribus terminis compoſito: reſolvendus eſt denominator in diviſores ſuos omnes primos. Et ſi diviſor ſit aliquis cui nullus alius eſt æqualis, Curva quadrari nequit: Sin duo vel plures ſint diviſores æquales, rejiciendus eſt eorum unus, & ſi adhuc alii duo vel plures ſint ſibi mutuo æquales & prioribus inæquales, rejiciendus eſt etiam eorum unus, & ſic in aliis omnibus æqualibus ſi adhuc plures ſint: deinde diviſor qui relinquitur vel contentum ſub diviſoribus omnibus qui relinquuntur, ſi plures ſunt, ponendum eſt pro R, & ejus quadrati reciprocum R^{-2} pro $R^{\lambda - 1}$, præterquam ubi contentum illud eſt quadratum vel cubus vel quadrato quadratum, &c. quo caſu ejus latus ponendum eſt pro R & poteſtatis index 2 vel 3 vel 4 negative ſumptus pro λ. et Ordinata ad denominatorem R^2 vel R^3 vel R^4 vel R^5 &c. reducenda.

Ut ſi Ordinata ſit $\frac{z^5+z^4-8z^3}{z^5+z^4-5z^3-z^2+8z-4}$; quoniam hæc fractio irreducibilis eſt & denominatoris diviſores ſunt pares, nempe $z-1$, $z-1$, $z-1$ & $z+2$, $z+2$, rejicio magnitudinis utriuſque diviſorem unum, & reliquorum $z-1$, $z-1$, $z+2$ contentum z^3-3z+2 pono pro R & ejus quadrati reciprocum $\frac{1}{R^2}$ ſeu R^{-2} pro $R^{\lambda-1}$. Dein Ordinatam ad denominatorem R^2 ſeu $R^{1-\lambda}$ reduco, & fit $\frac{z^6-9z^4+8z^3}{\overline{z^3-3z+2}|^2}$, i.e. $z^3\times\overline{8-9z+z^3}\times\overline{2-3z+z^3}|^{-2}$. Et inde eſt $a=8$, $b=-9$, $c=0$, $d=1$, &c. $e=2$, $f=-3$, $g=0$, $b=1$, $\lambda-1=-2$, $\lambda=-1$, $n=1$, $\theta-1=3$, $\theta=4=r$, $s=3$, $t=2$, $v=1$. Et his in ſerie ſcriptis prodit area $\frac{z^4}{z^3-3z+2}$, terminis omnibus in tota ſerie poſt primum evaneſcentibus.

Si denique Ordinata eſt fractio irreducibilis & ejus denominator contentum eſt ſub factore rationali $\mathcal{Q}$ & factore ſurdo irreducibili R^{π}, inveniendi ſunt lateris R diviſores omnes primi, & rejiciendus eſt diviſor unus magnitudinis cujuſque & per diviſores qui reſtant, ſiqui ſint, multiplicandus eſt factor rationalis $\mathcal{Q}$: & ſi factum æquale eſt lateri R vel lateris illius poteſtati alicui cujus index eſt numerus integer, eſto index ille m, & erit $\lambda-1=-\pi-m$, & $R^{\lambda-1}=R^{-\pi-m}$.

Ut ſi Ordinata ſit $\frac{3q^5-q^4x+9q^2x^2-q^2x^3-6x^3}{q^2-x^2\times\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, quoniam factoris ſurdi latus R ſeu $q^3+q^2x-qx^2-x^3$ diviſores habet $q+x$, $q+x$, $q-x$ qui duarum ſunt magnitudinum, rejicio diviſorem unum magnitudinis utriuſque & per diviſorem $q+x$ qui relinquitur, multiplico factorem rationalem q^2-x^2. Et quoniam factum $q^3+q^2x-qx^2-x^3$ æquale eſt lateri R, pono $m=1$, & inde, cum π ſit $\frac{1}{3}$, fit $\lambda-1=-\frac{4}{3}$. Ordinatam igitur reduco ad denominatorem $R^{-\frac{4}{3}}$, & fit

$$x^0\times\overline{3q^6+2q^5x+8q^4x^2+8q^3x^3-7q^2x^4-6qx^5}\times\overline{q^3+q^2x-qx^2-x^3}|^{-\frac{4}{3}}$$

Unde eſt $a=3q^6$, $b=2q^5$ &c. $e=q^3$, $f=q^2$ &c. $\theta-1=0$, $\theta=1=n$, $\lambda=-\frac{1}{3}$, $r=1$, $s=\frac{2}{3}$, $t=\frac{1}{3}$, $v=0$. Et his in ſerie ſcriptis prodit area $\frac{3q^2x+3x^3}{\overline{q^3+q^2x-qx^2-x^3}|^{\frac{1}{3}}}$, terminis omnibus in ſerie tota poſt tertium evaneſcentibus.

PROP. VI. THEOR. IV.

Si Curvæ abſciſſa AB ſit z, & ſcribantur R pro $e+fz^n+gz^{2n}+bz^{3n}+$&c. & S pro $k+lz^n+mz^{2n}+nz^{3n}+$&c. ſit autem Ordinatim applicata $z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ in $a+bz^n+cz^{2n}+dz^{3n}+$&c. et ſi terminorum, e, f, g, b, &c. et k, l, m, n, &c. rectangula ſint

$$\begin{array}{ccccc} ek & fk & gk & bk & \&c. \\ el & fl & gl & bl & \&c. \\ em & fm & gm & bm & \&c. \\ en & fn & gn & bn & \&c. \end{array}$$

Et ſi rectangulorum illorum coefficientes numerales ſint reſpective

$$\frac{\theta}{n}=r.\quad r+\lambda=s.\quad s+\lambda=t.\quad t+\lambda=v.\quad \&c.$$

$$r+\mu=s'.\quad s+\mu=t'.\quad t+\mu=v'.\quad v+\mu=w'.\quad \&c.$$

$$s'+\mu=t''.\quad t'+\mu=v''.\quad v'+\mu=w''.\quad w'+\mu=x''.\quad \&c.$$

$$t''+\mu=v'''.\quad v''+\mu=w'''.\quad w''+\mu=x'''.\quad x''+\mu=y'''.\quad \&c.$$

Area Curvæ erit hæc

$$z^{\theta}R^{\lambda}S^{\mu} \text{ in } +\frac{\frac{1}{n}a}{rek}$$

$$+\frac{\frac{1}{n}b\begin{array}{l}-sfk\\-s'el\end{array}A}{\overline{r+1}\times ek}z^{n}$$

$$+\frac{\frac{1}{n}c\begin{array}{l}-\overline{s+1}\times fk\\-s'+1\times el\end{array}B\begin{array}{l}-tgk\\-t'fl\\-t''em\end{array}A}{\overline{r+2}\times ek}z^{2n}$$

$$+\frac{\frac{1}{n}d\begin{array}{l}-\overline{s+2}\times fk\\-s'+2\times el\end{array}C\begin{array}{l}-\overline{t+1}\times gk\\-t'+1\times fl\\-t''+1\times em\end{array}B\begin{array}{l}-vhk\\-v'gl\\-v''fm\\-v'''en\end{array}A}{\overline{r+3}\times ek}z^{3n}$$

$$+\ \&c.$$

Ubi A denotat termini primi coefficientem datam $\frac{\frac{1}{n}a}{rek}$ cum ſigno ſuo + vel —, B coefficientem datam ſecundi, C coefficientem datam tertii, & ſic deinceps. Terminorum vero, a, b, c, &c. e, f, g, &c. k, l, m, &c. unus vel plures deeſſe poſſunt.

Demonſtratur Propoſitio ad modum præcedentis, & quæ ibi notantur hic obtinent. Pergit autem ſeries talium Propoſitionum in infinitum, & progreſſio ſeriei manifeſta eſt.

PROP. VII. THEOR. V.

Si pro $e+fz^{n}+gz^{2n}+$ &c. ſcribatur R ut ſupra, & in Curvæ alicujus Ordinata $z^{\theta\pm n\sigma}R^{\lambda\pm\tau}$ maneant quantitates datæ θ, n, λ, e, f, g, &c. et pro σ ac τ ſcribantur ſucceſſive numeri quicunque integri: & ſi detur Area unius ex Curvis quæ per Ordinatas innumeras ſic prodeuntes deſignantur ſi Ordinatæ ſunt duorum nominum in vinculo radicis, vel ſi dentur Areæ duarum ex Curvis ſi Ordinatæ ſunt trium nominum in vinculo radicis, vel Areæ trium ex Curvis ſi Ordinatæ ſunt quatuor nominum in vinculo radicis, & ſic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Pro nominibus hic habeo terminos omnes in vinculo radicis tam deficientes quam plenos quorum indices dignitatum ſunt in progreſſione arithmetica. Sic Ordinata $\sqrt{a^4 - ax^3 + x^4}$ ob terminos duos inter a^4 & $-ax^3$ deficientes pro quinquinomio haberi debet. At $\sqrt{a^4 + x^4}$ binomium eſt, & $\sqrt{a^4 + x^4 - \frac{x^8}{a^4}}$ trinomium, cum progreſſio jam per majores differentias procedat. Propoſitio vero ſic demonſtratur.

CAS. I.

Sunto Curvarum duarum Ordinatæ $pz^{\theta-1}R^{\lambda-1}$ & $qz^{\theta+n-1}R^{\lambda-1}$, & Areæ pA & qB, exiſtente R quantitate trium nominum $e + fz^n + gz^{2n}$. Et cum per Prop. 3. ſit $z^\theta R^\lambda$ area Curvæ cujus Ordinata eſt $\overline{\theta e + \overline{\theta + \lambda n} \times fz^n + \overline{\theta + 2\lambda n} \times gz^{2n}}$ in $z^{\theta-1}R^{\lambda-1}$, ſubduc Ordinatas & Areas priores de Area & Ordinata poſteriori, & manebit $\overline{\theta e - p + \overline{\underset{+\lambda n}{\theta} \times f - q} \times z^n + \underset{+2\lambda n}{\theta} \times gz^{2n}} \times z^{\theta-1}R^{\lambda-1}$ Ordinata nova Curvæ, et $\overline{z^\theta R^\lambda - pA - qB}$ ejuſdem Area. Pone $\theta e = p$, & $\theta f + \lambda nf = q$, & Ordinata evadet $\overline{\theta + 2\lambda n} \times gz^{2n} \times z^{\theta-1}R^{\lambda-1}$, & Area $z^\theta R^\lambda - \theta eA - \theta fB - \lambda nfB$. Divide utramque per $\theta g + 2\lambda ng$, & Aream prodeuntem dic C, & aſſumpta utcunque r, erit rC Area Curvæ cujus Ordinata eſt $rz^{\theta+2n-1}R^{\lambda-1}$. Et qua ratione ex Areis pA & qB Aream rC Ordinatæ $rz^{\theta+2n-1}R^{\lambda-1}$ congruentem invenimus, licebit ex Areis qB & rC Aream quartam puta sD, Ordinatæ $sz^{\theta+3n-1}R^{\lambda-1}$ congruentem invenire, & ſic deinceps in infinitum. Et par eſt ratio progreſſionis ab Areis B & A in partem contrariam pergentis. Si terminorum θ, $\theta + \lambda n$, & $\theta + 2\lambda n$ aliquis deficit & ſeriem abrumpit, aſſumatur Area pA in principio progreſſionis unius & Area qB in principio alterius, & ex his duabus Areis dabuntur Areæ omnes in progreſſione utraque. Et contra, ex aliis duabus Areis aſſumptis fit regreſſus per Analyſin ad Areas A & B, adeo ut ex duabus datis cæteræ omnes dentur. Q. E. O.

Hic eſt caſus Curvarum ubi ipſius z index θ augetur vel diminuitur perpetua additione vel ſubductione quantitatis n. Caſus alter eſt Curvarum ubi index λ augetur vel diminuitur unitatibus.

CAS. II.

Ordinatæ $pz^{\theta-1}R^\lambda$ et $qz^{\theta+n-1}R^\lambda$, quibus Areæ pA et qB jam reſpondeant, ſi in R ſeu $\overline{e + fz^n + gz^{2n}}$ ducantur ac deinde ad R viciſſim applicentur, evadunt $\overline{pe + pfz^n + pgz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $\overline{qez^n + qfz^{2n} + qgz^{3n}} \times z^{\theta-1}R^{\lambda-1}$. Et (per Prop. 3.) eſt $az^\theta R^\lambda$ area Curvæ cujus Ordinata eſt $\overline{\theta ae + \overline{\theta + \lambda n} \times afz^n + \overline{\theta + 2\lambda n} \times agz^{2n}} \times z^{\theta-1}R^{\lambda-1}$, et $bz^{\theta+n}R^\lambda$ area Curvæ cujus Ordinata eſt $\overline{\overline{\theta+n}\times bez^{n}+\overline{\theta+n+\lambda n}\times bfz^{2n}+\overline{\theta+n+2\lambda n}\times bgz^{3n}}\times z^{\theta-1}R^{\lambda-1}$.

Et harum quatuor Arearum ſumma eſt $pA+qB+az^{\theta}R^{\lambda}+bz^{\theta+n}R^{\lambda}$, & ſumma reſpondentium Ordinatarum

$$\begin{array}{llll} \overline{\theta ae+\overline{\theta+\lambda n}\times afz^{n}+\overline{\theta+2\lambda n}\times agz^{2n}+\overline{\theta+n+2\lambda n}\times bgz^{3n}}\times z^{\theta-1}R^{\lambda-1}. & & & \\ +pe+\overline{\theta+n}\times be & +\overline{\theta+n+\lambda n}\times bf & +1\times qg & \\ \quad +1\times pf & +1\times pg & & \\ \quad +1\times qe & +1\times qf & & \end{array}$$

Si terminus primus tertius & quartus ponantur ſeorſim æquales nihilo, per primum fiet $\theta ae+pe=0$, ſeu $-\theta a=p$, per quartum $-\theta b-nb-2\lambda nb=q$, & per tertium (eliminando p & q) $\frac{2ag}{f}=b$. Unde ſecundus fit $\frac{\lambda naff-4\lambda nage}{f}$, adeoq; ſumma quatuor Ordinatarum eſt $\frac{\lambda naff-4\lambda nage}{f}z^{\theta+n-1}R^{\lambda-1}$, & ſumma totidem reſpondentium Arearum eſt $az^{\theta}R^{\lambda}+\frac{2ag}{f}z^{\theta+n}R^{\lambda}-\theta aA+\frac{2\theta+2n+4\lambda n}{f}agB$. Dividantur hæ ſummæ per $\frac{\lambda naff-4\lambda nage}{f}$, & ſi Quotum poſterius dicatur D, erit D Area curvæ cujus Ordinata eſt Quotum prius $z^{\theta+n-1}R^{\lambda-1}$. Et eadem ratione ponendo omnes Ordinatæ terminos præter primum æquales nihilo poteſt Area Curvæ inveniri cujus Ordinata eſt $z^{\theta-1}R^{\lambda-1}$. Dicatur Area iſta C, & qua ratione ex Areis A & B inventæ ſunt Areæ C ac D, ex his Areis C ac D inveniri poſſunt aliæ duæ E & F Ordinatis $z^{\theta-1}R^{\lambda-2}$ et $z^{\theta+n-1}R^{\lambda-2}$ congruentes, & ſic deinceps in infinitum. Et per Analyſin contrariam regredi licet ab Areis E & F ad Areas C ac D, & inde ad Areas A & B, aliaſque quæ in progreſſione ſequuntur. Igitur ſi index λ perpetua unitatum additione vel ſubductione augeatur vel minuatur, & ex Areis quæ Ordinatis ſic prodeuntibus reſpondent duæ ſimpliciſſimæ habentur; dantur aliæ omnes in infinitum. Q. E. O.

CAS. III.

Et per caſus hoſce duos conjunctos, ſi tam index θ perpetua additione vel ſubductione ipſius n, quam index λ perpetua additione vel ſubductione unitatis, utcunque augeatur vel minuatur, dabuntur Areæ ſingulis prodeuntibus Ordinatis reſpondentes. Q. E. O.

CAS. IV.

Et ſimili argumento ſi Ordinata conſtat ex quatuor nominibus in vinculo radicali & dantur tres Arearum, vel ſi conſtat ex quinque nominibus & dantur quatuor Arearum, & ſic deinceps: dabuntur Areæ omnes quæ addendo vel ſubducendo numerum n indici θ vel unitatem indici λ generari poſſunt. Et par eſt ratio Curvarum ubi Ordinatæ ex binomiis conſtantur, & Area una earum quæ non ſunt Geometrice quadrabiles datur. Q. E. O.

PROP. VIII. THEOR. VI.

Si pro $e + fz^{n} + gz^{2n} +$ &c. et $k + lz^{n} + mz^{2n} +$ &c. ſcribantur R & S ut ſupra, & in Curvæ alicujus Ordinata $z^{\theta \pm n\sigma} R^{\lambda \pm \tau} S^{\mu \pm \upsilon}$ maneant quantitates datæ θ, n, λ, μ, e, f, g, k, l, m, &c. et pro σ, τ, & υ, ſcribantur ſucceſſive numeri quicunque integri: & ſi dentur Areæ duarum ex curvis quæ per Ordinatas ſic prodeuntes deſignantur ſi quantitates R & S ſunt binomia, vel ſi dentur Areæ trium ex curvis ſi R & S conjunctim ex quinque nominibus conſtant, vel Areæ quatuor ex curvis ſi R & S conjunctim ex ſex nominibus conſtant, & ſic deinceps in infinitum: dico quod dabuntur Areæ curvarum omnium.

Demonſtratur ad modum propoſitionis ſuperioris.

PROP. IX. THEOR. VII.

Æquantur Curvarum Areæ inter ſe quarum Ordinatæ ſunt reciproce ut fluxiones Abſciſſarum.

Nam contenta ſub Ordinatis & fluxionibus Abſciſſarum erunt æqualia, & fluxiones Arearum ſunt ut hæc contenta.

COROL. I.

Si aſſumatur relatio quævis inter Abſciſſas duarum Curvarum, & inde per Prop. 1. quæratur relatio fluxionum Abſciſſarum, & ponantur Ordinatæ reciproce proportionales fluxionibus, inveniri poſſunt innumeræ Curvæ quarum Areæ ſibi mutuo æquales erunt.

COROL. II.

Si enim Curva omnis cujus hæc eſt Ordinata $z^{\theta - 1} \times \overline{e + fz^{n} + gz^{2n} + \&c.}|^{\lambda}$, aſſumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu} = s$, & $z^{s} = x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $\frac{\nu}{n} x^{\frac{\nu\theta - n}{n}} \times \overline{e + fx^{\nu} + gx^{2\nu} + \&c.}|^{\lambda}$.

COROL.

COROL. III.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}$, aſſumendo quantitatem quamvis pro ν & ponendo $\frac{n}{\nu}=s$, & $z^{s}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$;

COROL. IV.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1}\times\overline{a+bz^{n}+cz^{2n}+\&c.}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}\times\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu}$, aſſumendo quantitatem quamvis pro ν, & ponendo $\frac{n}{\nu}=s$, & $z^{s}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt

$$\frac{\nu}{n}x^{\frac{\nu\theta-n}{n}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}\times\overline{k+lx^{\nu}+mx^{2\nu}+\&c.}|^{\mu}$$

COROL. V.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}$, ponendo $\frac{1}{z}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $\frac{1}{x^{\theta+1}}\times\overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}$ id eſt $\frac{1}{x^{\theta+1+n\lambda}}\times\overline{f+ex^{n}}|^{\lambda}$ ſi duo ſunt nomina in vinculo radicis, vel $\frac{1}{x^{\theta+1+2n\lambda}}\times\overline{g+fx^{n}+ex^{2n}}|^{\lambda}$ ſi tria ſunt nomina; & ſic deinceps.

COROL. VI.

Et Curva omnis cujus Ordinata eſt $z^{\theta-1}\times\overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}\times\overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu}$, ponendo $\frac{1}{z}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt

$$\frac{1}{x^{\theta+1}}\times\overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}\times\overline{k+lx^{-n}+mx^{-2n}+\&c.}|^{\mu}$$

id eſt $\frac{1}{x^{\theta+1+n\lambda+n\mu}}\times\overline{f+ex^{n}}|^{\lambda}\times\overline{l+kx^{n}}|^{\mu}$ ſi bina ſunt nomina in vinculis radicum, vel $\frac{1}{x^{\theta+1+2n\lambda+n\mu}}\times\overline{g+fx^{n}+ex^{2n}}|^{\lambda}\times\overline{l+kx^{n}}|^{\mu}$ ſi tria ſunt nomina in vinculo radicis prioris ac duo in vinculo poſterioris: & ſic in aliis.

Et nota quod Areæ duæ æquales in noviſſimis hiſce duobus Corollariis jacent ad contrarias partes Ordinatarum. Si Area in alterutra Curva adjacet Abſciſſæ, Area huic æqualis in altera Curva adjacet Abſciſſæ productæ.

C O R O L. VII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definiatur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha} \times \overline{e + fy^{n}z^{\delta} + gy^{2n}z^{2\delta} + hy^{3n}z^{3\delta} + \&c.} = z^{\beta} \times \overline{k + ly^{n}z^{\delta} + my^{2n}z^{2\delta} + \&c.}$$

hæc Figura aſſumendo $s = \frac{n-\delta}{n}$, $x = \frac{1}{s}z^{s}$, & $\lambda = \frac{n-\delta}{\alpha\delta + \beta n}$, migrat in aliam ſibi æqualem cujus Abſciſſa x, ex data Ordinata v, determinatur per æquationem non affectam $\frac{1}{s}v^{\alpha\lambda} \times \overline{e + fv^{n} + gv^{2n} + \&c.}\big|^{\lambda} \times \overline{k + lv^{n} + mv^{2n} + \&c.}\big|^{-\lambda} = x$.

C O R O L. VIII.

Si relatio inter Curvæ alicujus Ordinatam y & Abſciſſam z definitur per æquationem quamvis affectam hujus formæ,

$$y^{\alpha} \times \overline{e + fy^{n}z^{\delta} + gy^{2n}z^{2\delta} + \&c.} = z^{\beta} \times \overline{k + ly^{n}z^{\delta} + my^{2n}z^{2\delta} + \&c.}$$
$$+ z^{\gamma} \times \overline{p + qy^{n}z^{\delta} + ry^{2n}z^{2\delta} + \&c.}$$

hæc Figura aſſumendo $s = \frac{n-\delta}{n}$, $x = \frac{1}{s}z^{s}$, $\mu = \frac{\alpha\delta + \beta n}{n-\delta}$, & $\nu = \frac{\alpha\delta + \gamma n}{n-\delta}$, migrat in aliam ſibi æqualem cujus Abſciſſa x ex data Ordinata v determinatur per æquationem minus affectam $v^{\alpha} \times \overline{e + fv^{n} + gv^{2n} + \&c.} = s^{\mu}x^{\mu} \times \overline{k + lv^{n} + mv^{2n} + \&c.}$
$+ s^{\nu}x^{\nu} \times \overline{p + qv^{n} + rv^{2n} + \&c.}$

C O R O L. IX.

Curva omnis cujus Ordinata eſt $\pi z^{\theta - 1} \times \overline{\nu e + \overline{\nu + n}fz^{n} + \overline{\nu + 2n}gz^{2n} + \&c.} \times \overline{e + fz^{n} + gz^{2n} + \&c.}\big|^{\lambda - 1}$ in $\overline{a + b \times \overline{ez^{\nu} + fz^{\nu + n} + gz^{\nu + 2n} + \&c.}\big|^{\tau}}\Big|^{\omega}$, ſi ſit $\theta = \lambda\nu$, & aſſumantur $x = \overline{ez^{\nu} + fz^{\nu + n} + gz^{\nu + 2n} + \&c.}\big|^{\pi}$, $\sigma = \frac{\tau}{\pi}$, & $\vartheta = \frac{\lambda - \pi}{\pi}$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta} \times \overline{a + bx^{\sigma}}\big|^{\omega}$. Et nota quod Ordinata prior in hoc Corollario evadit ſimplicior ponendo $\lambda = 1$, vel ponendo $\tau = 1$, & efficiendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel etiam ponendo $\omega = -1$, & $\lambda = 1 = \tau = \sigma = \pi$, ut alios caſus prætercam.

COROL.

COROL. X.

Pro $\overline{ez^{\nu}+fz^{\nu+\eta}+gz^{\nu+2\eta}+\&c.}$ $\overline{\nu ez^{\nu-1}+\overline{\nu+\eta}fz^{\nu+\eta-1}+\overline{\nu+2\eta}gz^{\nu+2\eta-1}+\&c.}$ $\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}$ et $\overline{\eta lz^{\eta-1}+2\eta mz^{2\eta-1}+\&c.}$ ſcribantur R, $\dot{r}$, S & $\dot{s}$ reſpective, & Curva omnis cujus Ordinata eſt $\overline{\pi S\dot{r}+\varphi R\dot{s}}\times R^{\lambda-1}S^{\mu-1}\times\overline{aS^{\upsilon}+bR^{\tau}}|^{\omega}$, ſi ſit $\frac{\mu-\upsilon\omega}{\lambda}=\frac{\upsilon}{\tau}=\frac{\varphi}{\pi}$, $\frac{\tau}{\pi}=\sigma$, $\frac{\lambda-\pi}{\pi}=\vartheta$, & $R^{\pi}S^{\varphi}=x$, migrat in aliam ſibi æqualem cujus Ordinata eſt $x^{\vartheta}\times\overline{a+bx^{\sigma}}|^{\omega}$. Et nota quod Ordinata prior evadit ſimplicior, ponendo unitates pro τ, υ, & λ vel μ, & faciendo ut radix dignitatis extrahi poſſit cujus index eſt ω, vel ponendo $\omega=-1$ vel $\mu=0$.

PROP. X. PROB. III.

Invenire Figuras ſimpliciſſimas cum quibus Curva quævis Geometrice comparari poteſt, cujus Ordinatim applicata y *per æquationem non affectam ex data Abſciſſa* z *determinatur.*

CAS. I.

Sit Ordinata $az^{\theta-1}$, & Area erit $\frac{1}{\theta}az^{\theta}$, ut ex Prop. 5. ponendo $b=0=c=d=f=g=h$, & $e=1$, facile colligitur.

CAS. II.

Sit Ordinata $az^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$, et ſi Curva cum Figuris rectilineis Geometrice comparari poteſt, quadrabitur per Prop. 5. ponendo $b=0=c=d$. Sin minus convertetur in aliam Curvam ſibi æqualem cujus Ordinata eſt $\frac{a}{\eta}x^{\frac{\theta-\eta}{\eta}}\times\overline{e+fx+gx^{2}+\&c.}|^{\lambda-1}$ per Corol. 2, Prop. 9. Deinde ſi de dignitatum indicibus $\frac{\theta-\eta}{\eta}$ & $\lambda-1$ per Prop. 7. rejiciantur unitates donec dignitates illæ ſiant quam minimæ, devenietur ad Figuras ſimpliciſſimas quæ hac ratione colligi poſſunt. Dein harum unaquæque per Corol. 5, Prop. 9. dat aliam quæ nonnunquam ſimplicior eſt. Et ex his per Prop. 3. & Corol. 9, & 10, Prop. 9. inter ſe collatis, Figuræ adhuc ſimpliciores quandoque prodeunt. Denique ex Figuris ſimpliciſſimis aſſumptis facto regreſſu computabitur Area quæſita.

C A S. III.

Sit Ordinata $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$, & hæc Figura ſi quadrari poteſt, quadrabitur per Prop. 5. Sin minus, diſtinguenda eſt Ordinata in partes $z^{\theta-1}\times a\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$, $z^{\theta-1}\times bz^{\eta}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$, &c. et per Caſ. 2. inveniendæ ſunt Figuræ ſimpliciſſimæ cum quibus Figuræ partibus illis reſpondentes comparari poſſunt.

Nam Areæ figurarum partibus illis reſpondentium ſub ſignis ſuis + & — conjunctæ component Aream totam quæſitam.

C A S. IV.

Sit Ordinata $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda-1}$ in $\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}|^{\mu-1}$; et ſi Curva quadrari poteſt, quadrabitur per Prop. 6. Sin minus, convertetur in ſimpliciorem per Corol. 4, Prop. 9. ac inde comparabitur cum Figuris ſimpliciſſimis per Prop. 8. et Corol. 6, 9 & 10, Prop 9. ut fit in Caſu 2 & 3.

C A S. V.

Si Ordinata ex variis partibus conſtat, partes ſingulæ pro Ordinatis curvarum totidem habendæ ſunt, & Curvæ illæ quotquot quadrari poſſunt, ſigillatim quadrandæ ſunt, earumque Ordinatæ de Ordinata tota demendæ.

Dein Curva quam Ordinatæ pars reſidua deſignat ſeorſim (ut in Caſu 2, 3 & 4,) cum Figuris ſimpliciſſimis comparanda eſt cum quibus comparari poteſt. Et ſumma Arearum omnium pro Area Curvæ propoſitæ habenda eſt.

COROL. I.

Hinc etiam Curva omnis cujus Ordinata eſt radix quadratica affecta æquationis ſuæ, cum Figuris ſimpliciſſimis ſeu rectilineis ſeu curvilineis comparari poteſt. Nam radix illa ex duabus partibus ſemper conſtat quæ ſeorſim ſpectatæ non ſunt æquationum radices affectæ.

Proponatur æquatio $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$, & extracta radix erit $y = \frac{a^3 + z^3 + a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$, cujus pars rationalis $\frac{a^3 + z^3}{aa + zz}$ & pars irrationalis $\frac{a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$ ſunt Ordinatæ curvarum quæ per hanc Propoſitionem vel quadrari poſſunt vel cum Figuris ſimpliciſſimis comparari cum quibus collationem Geometricam admittunt.

COROL. II.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 7. Prop. 9. in æquationem non affectam migrat, vel quadratur per hanc Propoſitionem ſi quadrari poteſt, vel comparatur cum Figuris ſimpliciſſimis cum quibus comparari poteſt. Et hac ratione Curva omnis quadratur cujus æquatio eſt trium terminorum. Nam æquatio illa ſi affecta ſit, transmutatur in non affectam per Corol. 7. Prop. 9. ac deinde per Corol. 2. & 5. Prop. 9. in ſimpliciſſimam migrando, dat vel quadraturam Figuræ ſi quadrari poteſt, vel Curvam ſimpliciſſimam quacum comparatur.

COROL. III.

Et Curva omnis cujus Ordinata per æquationem quamvis affectam definitur quæ per Corol. 8. Prop. 9. in æquationem quadraticam affectam migrat; vel quadratur per hanc Propoſitionem & hujus Corol. 1. ſi quadrari poteſt, vel comparatur cum Figuris ſimpliciſſimis cum quibus collationem Geometricam admittit.

SCHOLIUM.

Ubi quadrandæ ſunt Figuræ; ad Regulas haſce generales ſemper recurrere nimis moleſtum eſſet: præſtat Figuras quæ ſimpliciores ſunt & magis uſui eſſe poſſunt ſemel quadrare & quadraturas in Tabulam referre, deinde Tabulam conſulere quoties ejuſmodi Curvam aliquam quadrare oportet. Hujus autem generis ſunt Tabulæ duæ ſequentes, in quibus z denotat Abſciſſam, y Ordinatam rectangulam, & t Aream Curvæ quadrandæ, & d, e, f, g, h, n ſunt quantitates datæ cum ſignis ſuis $+$ & $-$.

TABULA

CURVARUM SIMPLICIORUM QUÆ QUADRARI POSSUNT

Curvarum Formæ			Curvarum Areæ
I		$dz^{n-1} = y$	$\frac{d}{n}z^{n} = t$
II		$\frac{dz^{n-1}}{e^2 + 2efz^n + f^2z^{2n}} = y$	$\frac{dz^n}{ne^2 + nefz^n} = t$, vel $\frac{-d}{nef + nf^2z^n} = t$
III	1	$dz^{n-1}\sqrt{e+fz^n} = y$	$\frac{2d}{3nf}R^3 = t$, exiſtente $R = \sqrt{e+fz^n}$
	2	$dz^{2n-1}\sqrt{e+fz^n} = y$	$\frac{-4e+6fz^n}{15nf^2}dR^3 = t$
	3	$dz^{3n-1}\sqrt{e+fz^n} = y$	$\frac{16e^2 - 24efz^n + 30f^2z^{2n}}{105nf^3}dR^3 = t$
	4	$dz^{4n-1}\sqrt{e+fz^n} = y$	$\frac{-96e^3 + 144e^2fz^n - 180ef^2z^{2n} + 210f^3z^{3n}}{945nf^4}dR^3 = t$
IV	1	$\frac{dz^{n-1}}{\sqrt{e+fz^n}} = y$	$\frac{2d}{nf}R = t$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-4e+2fz^n}{3nf^2}dR = t$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n}} = y$	$\frac{16e^2 - 8efz^n + 6f^2z^{2n}}{15nf^3}dR = t$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n}} = y$	$\frac{-96e^3 + 48e^2fz^n - 36ef^2z^{2n} + 30f^3z^{3n}}{105nf^4}dR = t$

John Senex ſculpt.

TABU

CURVARUM SIMPLICIORUM QUÆ CUM El

Sit jam aGD vel PGD vel GDS Sectio Conica cujus Area ad Quadratur
Semiaxis conjugatus AP, datum Abſciſſæ principium A vel a vel α, Abſci
vel aBDG vel αBDG $= s$, exiſtente αG Ordinata ad punctum α. Junguntur K
parallelogrammum ABDO. Et ſiquando ad quadraturam Curvæ propoſitæ requiru
Ordinata τ, et Area σ. Sit autem $\div$ differentia duarum quantitatum ubi incert
Et in Forma ſexta ſcribatur p pro $\sqrt{ff-4eg}$.

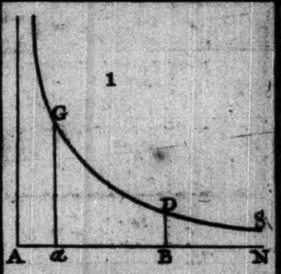
Curvarum Formæ			Sectionis Conicæ	
			Abſciſſa	Ordinata
I.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}}=y$	$z^{\eta}=x$	$\frac{d}{e+fx}=v$
II	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
	3	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$
III	1	$\frac{d}{z}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
IV	1	$\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^2$	$\sqrt{f+ex^2}=v$
		vel sic	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^2}=v$

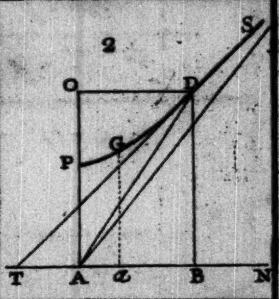
ULA

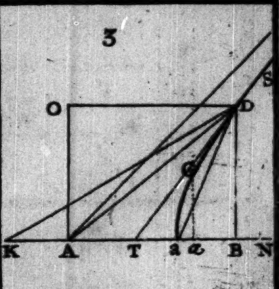
M EILIPSI ET HYPERBOLA COMPARARI POSSUNT.

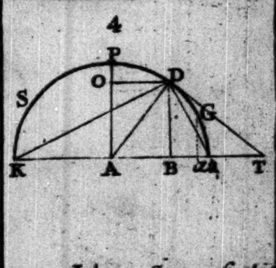
adraturam Curvæ propoſitæ requiritur, ſitq; ejus Centrum A, Axis Ka, Vertex a,
α, Abſciſsa AB vel aB vel αB = x, Ordinata rectangula BD = v, et Area ABDP
antur KD, AD, aD; ducatur Tangens DT occurrens Abſciſsæ AB in T, & compleatur
requiruntur Areæ duarum Sectionum Conicarum, dicatur poſterioris Abſciſsa ξ,
i incertum eſt utrum poſterior de priori an prior de poſteriori ſubduci debeat.

	CURVARUM AREÆ
	$\frac{1}{\eta}s = t = \frac{\alpha GDB}{\eta}$. *Fig.* 1.
	$\frac{d}{\eta f}z^{\eta} - \frac{e}{\eta f}s = t$.
	$\frac{d}{2\eta f}z^{2\eta} - \frac{de}{\eta f^2}z^{\eta} + \frac{e^2}{\eta f^2}s = t$.
	$\frac{2xv \div 4s}{\eta} = t = \frac{4}{\eta}$ADGa . *Fig.* 3. 4.
	$\frac{2d}{\eta f}z^{\frac{1}{2}\eta} + \frac{4es - 2exv}{\eta f} = t$.
	$\frac{2d}{3\eta f}z^{\frac{3}{2}\eta} - \frac{2de}{\eta f^2}z^{\frac{1}{2}\eta} + \frac{2e^2xv - 4e^2s}{\eta f^2} = t$.
	$\frac{4de}{\eta f} \times \overline{\frac{v^3}{2ex} - s} = t = \frac{4de}{\eta f}$ in aGDT, *vel in* APDB ÷ TDB *Fig.* 2. 3. 4.
	$\frac{8de^2}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e} \div \frac{f^2v}{4e^2x}} = t = \frac{8de^2}{\eta f^2}$ in aGDA $+ \frac{f^2v}{4e^2x}$. *Fig.* 2. 3. 4.
	$-\frac{2d}{\eta}s = t = \frac{2d}{\eta}$APDB ſeu $\frac{2d}{\eta}$ aGDB . *Fig.* 2. 3. 4.
	$\frac{4de}{\eta f} \times \overline{s - \frac{1}{2}xv - \frac{fv}{2e}} = t = \frac{4de}{\eta f} \times$ aGDK . *Fig.* 3. 4.
	$-\frac{d}{\eta}s = t = \frac{d}{\eta} \times -$ aGDB *vel* BDPK . *Fig.* 4.
	$\frac{3dfs - 2dv^3}{6\eta e} = t$.
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in PAD *vel in* aGDA . *Fig.* 2 3. 4.
	$\frac{8de}{\eta f^2} \times \overline{s - \frac{1}{2}xv - \frac{fv}{4e}} = t = \frac{8de}{\eta f^2}$ in aGDA . *Fig.* 3. 4.
	$\frac{2d}{\eta e} \times \overline{s - xv} = t = \frac{2d}{\eta e}$ in POD *vel in* AODGa . *Fig.* 2. 3. 4.
	$\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = t = \frac{4d}{\eta f}$ in aDGa . *Fig.* 3. 4.
	$\frac{-d}{\eta e} \times \overline{3s \div 2xv} = t = \frac{d}{\eta e}$ in 3aDGa ÷ Δ aDB . *Fig.* 2. 3. 4.
	$\frac{10dfxv - 15dfs - 2dex^2v}{6\eta e^2} = t$









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RESIDUUM TABULÆ CURVARUM SIMPLICIORUM

CURVARUM FORMÆ			SECTIONIS CONICÆ	
			Abſciſsa	Ordinata
V	1	$\frac{dz^{n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$
	1	vel sic	$\sqrt{\frac{dz^{2n}}{e+fz^n+gz^{2n}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^2-4eg}{4e^2}x^2}=v$
	2	$\frac{dz^{2n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{d}{e+fz^n+gz^{2n}}}=x$ $fz^n+gz^{2n}=\xi$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$ $\frac{1}{e+\xi}=\Upsilon$
VI	1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^n}}=x$ $\sqrt{\frac{2dg}{f+p+2gz^n}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^2}=v$ $\sqrt{d+\frac{-f-p}{2g}\xi^2}=\Upsilon$
	2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n+gz^{2n}}=y$	$\sqrt{\frac{2dez^n}{fz^n-pz^n+2e}}=x$ $\sqrt{\frac{2dez^n}{fz^n+pz^n+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^2}=v$ $\sqrt{d+\frac{-f-p}{2e}\xi^2}=\Upsilon$
VII	1	$\frac{d}{z}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$ $\frac{1}{z^n}=\xi$	$\sqrt{e+fx+gx^2}=v$ $\sqrt{g+f\xi+e\xi^2}=\Upsilon$
	2	$dz^{n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	3	$dz^{2n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	4	$dz^{3n-1}\sqrt{e+fz^n+gz^{2n}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
VIII	1	$\frac{dz^{n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n+gz^{2n}}}=y$	$z^n=x$	$\sqrt{e+fx+gx^2}=v$
IX	1	$\frac{dz^{n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
	2	$\frac{dz^{2n-1}\sqrt{e+fz^n}}{g+hz^n}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
X	1	$\frac{dz^{n-1}}{g+hz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
	2	$\frac{dz^{2n-1}}{g+hz^n\sqrt{e+fz^n}}=y$	$\sqrt{\frac{d}{g+hz^n}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$
XI	1	$dz^{-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$ $\sqrt{h+gz^{-n}}=\xi$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$ $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^2}=\Upsilon$
	2	$dz^{n-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$
	3	$dz^{2n-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$	$\sqrt{g+hz^n}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$

UM QUÆ CUM ELLIPSI ET HYPERBOLA COMPARARI POSSUNT

CURVARUM AREÆ

$$\frac{xv-2s}{\eta}=t.$$

$$\frac{2s-xv}{\eta}=t.$$

$$\frac{d\sigma+2fs-fxv}{2\eta g}=t.$$

$$\frac{2xv-4s-2\xi\tau+4\sigma}{\eta p}=t.$$

$$\frac{4s-2xv-4\sigma+2\xi\tau}{\eta p}=t.$$

$$\frac{4de^2\xi\tau+2def\tau-2dfgxv\substack{+4deg\\-2ffd}v-8de^2\sigma+4dfgs}{4\eta eg-\eta ff}=t.$$

$\frac{d}{\eta}s=t=\frac{d}{\eta}\times\alpha GDB$. Fig . 2.3.4.

$$\frac{d}{3\eta g}v^3-\frac{df}{2\eta g}s=t.$$

$$\frac{6dgx-5df}{24\eta g^2}v^3+\frac{5df^2-4deg}{16\eta g^2}s=t.$$

$\frac{8dgs-4dgxv-2dfv}{4\eta eg-\eta f^2}=t=\frac{8dg}{4\eta eg-\eta f^2}\times\alpha GDB\pm\Delta DBA$. Fig. 2.4.

$$\frac{-4dfs+2dfxv+4dev}{4\eta eg-\eta f^2}=t.$$

$$\frac{\substack{3dff\\-4deg}s\substack{-2dff\\+4deg}xv-2defv}{4\eta eg^2-\eta f^2g}=t.$$

$$\frac{\substack{36defg\\-15df^3}s\substack{+8degg\\-2df^2g}x^2v\substack{+10dfff\\-28defg}xv\substack{+10deff\\-16de^2g}v}{24\eta eg^3-6\eta f^2g^2}=t.$$

$$\frac{\substack{4fg\\-4eh}s\substack{-2fg\\+2eh}xv+2df\frac{v}{x}}{\eta fh}=t.$$

$$\frac{\substack{4egh\\-4fgg}s\substack{-2egh\\+2fgg}xv+\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{\eta fh^2}=t.$$

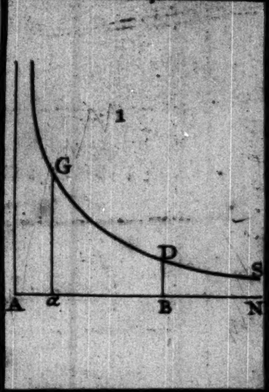
$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}ADGa$. Fig. 3.4.

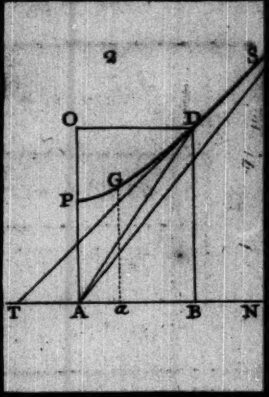
$$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t.$$

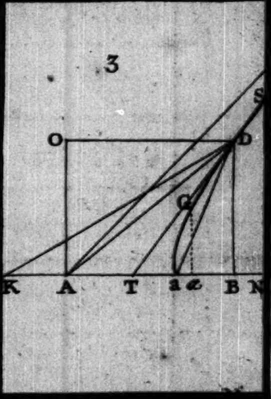
$$\frac{2dxv^3z^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}=t.$$

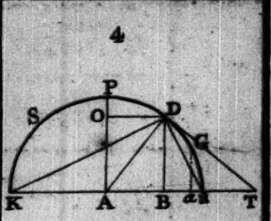
$$\frac{2d}{\eta h}s=t.$$

$$\frac{dhxv^3\substack{-3dfg\\-deh}s}{2\eta fh^2}=t.$$









Iohan. Senex sculp.

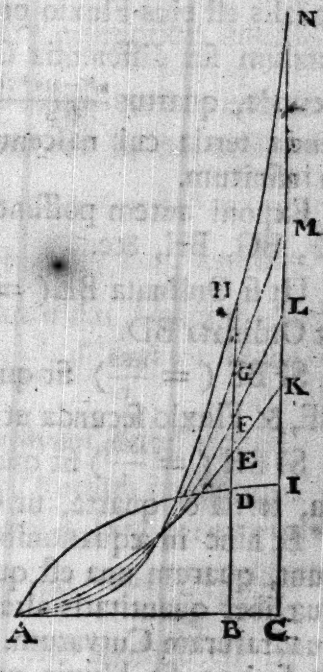
In Tabulis hiſce, ſeries Curvarum cujuſque formæ utrinque in infinitum continuari poteſt. Scilicet in Tabula prima, in numeratoribus Arearum formæ tertiæ & quartæ, numeri coefficientes initialium terminorum (2, —4, 16, —96, 868, &c.) generantur multiplicando numeros — 2, —4, —6, —8, —10, &c. in ſe continuo, & ſubſequentium terminorum coefficientes ex initialibus derivantur multiplicando ipſos gradatim, in Forma quidem tertia, per $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. in quarta vero per $-\frac{1}{2}$ $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c. Et Denominatorum coefficientes 3, 15, 105, &c. prodeunt multiplicando numeros 1, 3, 5, 7, 9, &c. in ſe continue.

In ſecunda vero Tabula, ſeries Curvarum formæ primæ, ſecundæ, quintæ, ſextæ, nonæ, & decimæ ope ſolius diviſionis, & formæ reliquæ ope Propoſitionis tertiæ & quartæ, utrinque producuntur in infinitum.

Quinetiam hæ ſeries mutando ſignum numeri n variari ſolent. Sic enim *e. g.* Curva $\frac{d}{z}\sqrt{e+fz^n}=y$, evadit $\frac{d}{z^{\frac{1}{2}n+1}}\sqrt{f+ez^n}=y$.

PROP. XI. THEOR. VIII.

Sit ADIC Curva quævis Abſciſſam habens AB $=z$, & Ordinatam BD$=y$, & ſit AEKC Curva alia cujus Ordinata BE æqualis eſt prioris areæ ADB ad unitatem applicatæ, & AFLC Curva tertia cujus Ordinata BF æqualis eſt ſecundæ Areæ AEB ad unitatem applicatæ, & AGMC Curva quarta cujus Ordinata BG æqualis eſt tertiæ areæ AFB ad unitatem applicatæ, & AHNC Curva quinta cujus Ordinata BH æqualis eſt quartæ Areæ AGB ad unitatem applicatæ, & ſic deinceps in infinitum. Et ſunto A, B, C, D, E, &c. Areæ Curvarum Ordinatas habentium y, zy, z^2y, z^3y, z^4y, &c. et Abſciſſam communem z.



Detur Abſciſſa quævis AC $=t$, ſitque BC $=t-z=x$, & ſunto P, Q, R, S, T Areæ Curvarum Ordinatas habentium y, xy, x^2y, x^3y, x^4y &c. et Abſciſſam communem x.

Terminentur autem hæ Areæ omnes ad Abſciſſam totam datam AC, nec non ad Ordinatam poſitione datam & infinite productam CI:

Et erit Arearum ſub initio poſitarum

Prima ADIC $=A=P$,
Secunda AEKC $=tA-B=Q$.
Tertia AFLC $=\frac{t^2A-2tB+C}{2}=\frac{1}{2}R$.
Quarta AGMC $=\frac{t^3A-3t^2B+3tC-D}{6}=\frac{1}{6}S$.
Quinta AHNC $=\frac{t^4A-4t^3B+6t^2C-4tD+E}{24}=\frac{1}{24}T$.

COROL.

COROL.

Unde ſi Curvæ quarum Ordinatæ ſunt y, zy, z^2y, z^3y, &c. vel y, xy, x^2y, x^3y, &c. quadrari poſſunt, quadrabuntur etiam Curvæ ADIC, AEKC, AFLC, AGMC, &c. et habebuntur Ordinatæ BE, BF, BG, BH, Areis Curvarum proportionales.

SCHOLIUM.

Quantitatum fluentium Fluxiones eſſe primas, ſecundas, tertias, quartas, aliaſque diximus ſupra. Hæ Fluxiones ſunt ut termini ſerierum infinitarum convergentium.

Ut ſi z^n ſit quantitas fluens & fluendo evadat $\overline{z+o}|^n$, deinde reſolvatur in ſeriem convergentem $z^n + noz^{n-1} + \frac{nn-n}{2}ooz^{n-2} + \frac{n^3-3nn+2n}{6}o^3z^{n-3} +$ &c. terminus primus hujus ſeriei z^n erit quantitas illa fluens, ſecundus noz^{n-1} erit ejus incrementum primum ſeu differentia prima cui naſcenti proportionalis eſt ejus Fluxio prima, tertius $\frac{nn-n}{2}ooz^{n-2}$ erit ejus incrementum ſecundum ſeu differentia ſecunda cui naſcenti proportionalis eſt ejus Fluxio ſecunda, quartus $\frac{n^3-3nn+2n}{6}o^3z^{n-3}$ erit ejus incrementum tertium ſeu differentia tertia cui naſcenti Fluxio tertia proportionalis eſt, & ſic deinceps in infinitum.

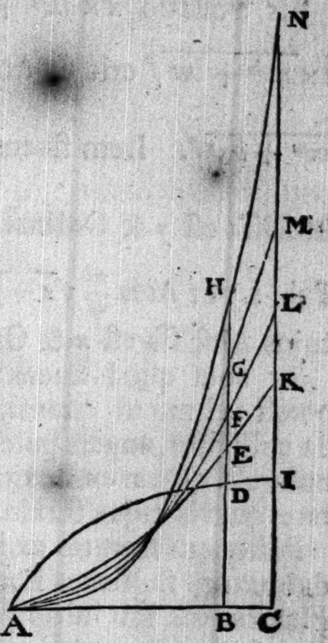
Exponi autem poſſunt hæ Fluxiones per Curvarum Ordinatas BD, BE, BF, BG, BH, &c.

Ut ſi Ordinata BE $\left(=\frac{ADB}{1}\right)$ ſit quantitas fluens, erit ejus Fluxio prima ut Ordinata BD.

Si BF $\left(=\frac{AEB}{1}\right)$ ſit quantitas fluens, erit ejus Fluxio prima ut Ordinata BE, & Fluxio ſecunda ut Ordinata BD.

Si BH $\left(=\frac{AGB}{1}\right)$ ſit quantitas fluens, erunt ejus Fluxiones prima, ſecunda, tertia & quarta, ut Ordinatæ BG, BF, BE, BD reſpective.

Et hinc in æquationibus quæ quantitates tantum duas incognitas involvunt, quarum una eſt quantitas uniformiter fluens & altera eſt Fluxio quælibet quantitatis alterius fluentis, inveniri poteſt fluens illa altera per quadraturam Curvarum. Exponatur enim Fluxio ejus per Ordinatam BD, & ſi hæc ſit Fluxio prima, quæratur Area ADB $=$ BE $\times 1$, ſi Fluxio ſecunda, quæratur Area AEB $=$ BF $\times 1$, ſi Fluxio tertia, quæratur Area AFB $=$ BG $\times 1$, &c. et Area inventa erit exponens fluentis quæſitæ.



Sed & in æquationibus quæ fluentem & ejus fluxionem primam ſine altera fluente, vel duas ejuſdem fluentis fluxiones, primam & ſecundam, vel ſecundam & tertiam, vel tertiam & quartam, &c. ſine alterutra fluente involvunt: inveniri poſſunt fluentes per quadraturam Curvarum. Sit æquatio $aa\dot{v}=av+vv$, exiſtente $v=BE$, $\dot{v}=BD$, $z=AB$, & $\dot{z}=1$, & æquatio illa complendo dimenſiones Fluxionum, evadet $aa\dot{v}=av\dot{z}+vv\dot{z}$, ſeu $\frac{aa\dot{v}}{av+vv}=\dot{z}$.

Jam fluat v uniformiter, & ſit ejus Fluxio $\dot{v}=1$, & erit $\frac{aa}{av+vv}=\dot{z}$, & quadrando Curvam cujus Ordinata eſt $\frac{aa}{av+vv}$ & Abſciſſa v, habebitur fluens z. Adhæc ſit æquatio $aa\ddot{v}=a\dot{v}+\dot{v}\dot{v}$, exiſtente $v=BF$, $\dot{v}=BE$, $\ddot{v}=BD$, & $z=AB$, & per relationem inter $\ddot{v}$ & $\dot{v}$ ſeu BD & BE invenietur relatio inter AB & BE, ut in exemplo ſuperiore. Deinde per hanc relationem invenietur relatio inter AB & BF quadrando Curvam AEB.

Æquationes quæ tres incognitas quantitates involvunt aliquando reduci poſſunt ad æquationes quæ duas tantum involvunt, & in his caſibus fluentes invenientur ex fluxionibus ut ſupra. Sit æquatio $a-bx^m=cxy^n\dot{y}+dy^{2n}\dot{y}\dot{y}$, ponatur $y^n\dot{y}=\dot{v}$, & erit $a-bx^m=cx\dot{v}+d\dot{v}\dot{v}$. Hæc æquatio quadrando Curvam cujus Abſciſſa eſt x & Ordinata $\dot{v}$ dat Aream v; & æquatio altera $y^n\dot{y}=\dot{v}$, regrediendo ad fluentes, dat $\frac{1}{n+1}y^{n+1}=v$: Unde habetur fluens y.

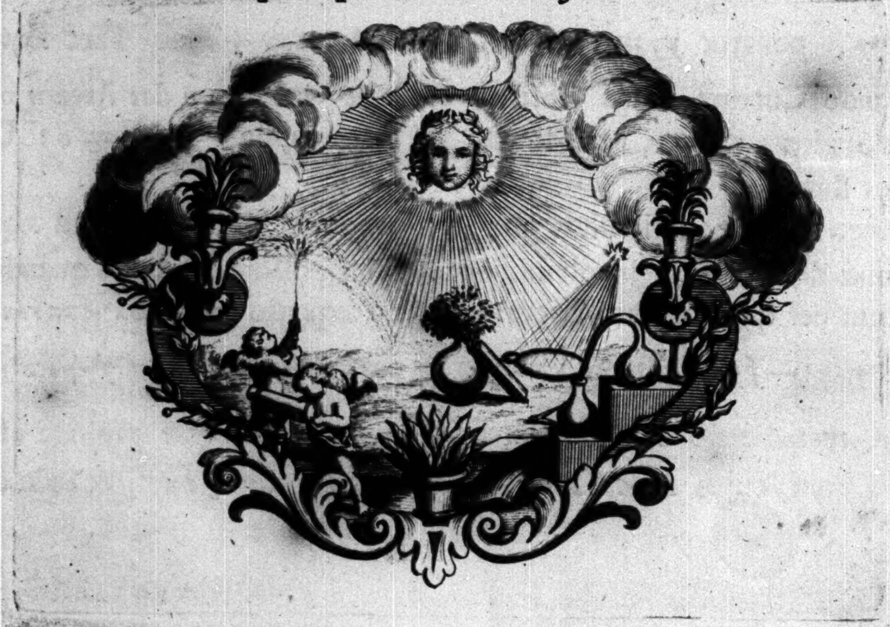
Quinetiam in æquationibus, quæ tres incognitas involvunt, & ad æquationes quæ duas tantum involvunt, reduci non poſſunt, fluentes quandoque prodeunt per quadraturam Curvarum. Sit æquatio $\overline{ax^m+bx^n}|^p=rex^{r-1}y^s+sex^r\dot{y}y^{s-1}-f\dot{y}y^t$, exiſtente $\dot{x}=1$; Et pars poſterior $rex^{r-1}y^s+sex^r\dot{y}y^{s-1}-f\dot{y}y^t$, regrediendo ad fluentes, fit $ex^ry^s-\frac{f}{t+1}y^{t+1}$, quæ proinde eſt ut Area Curvæ cujus Abſciſſa eſt x & Ordinata $\overline{ax^m+bx^n}|^p$, & inde datur fluens y.

Sit æquatio $\dot{x}\times\overline{ax^m+bx^n}|^p=\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$, Et fluens cujus Fluxio eſt $\dot{x}\times\overline{ax^m+bx^n}|^p$ erit ut Area Curvæ cujus Abſciſſa eſt x & Ordinata eſt $\overline{ax^m+bx^n}|^p$. Item fluens cujus Fluxio eſt $\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ erit ut Area Curvæ cujus Abſciſſa eſt y & Ordinata $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$, id eſt (per Caſum 1. Formæ quartæ Tab. I.) ut Area $\frac{2d}{nf}\sqrt{e+fy^n}$. Pono ergo $\frac{2d}{nf}\sqrt{e+fy^n}$ æqualem Areæ Curvæ cujus Abſciſſa eſt x & Ordinata $\overline{ax^m+bx^n}|^p$, & habebitur fluens y.

Et nota quod fluens omnis, quæ ex Fluxione prima colligitur, augeri poteſt vel minui quantitate quavis non fluente. Quæ ex Fluxione ſecunda colligitur, augeri poteſt vel minui quantitate quavis cujus Fluxio ſecunda nulla eſt. Quæ ex fluxione tertia colligitur, augeri poteſt vel minui quantitate quavis cujus fluxio tertia nulla eſt. Et ſic deinceps in infinitum.

Poſtquam fluentes ex fluxionibus collectæ ſunt, ſi de veritate Concluſionis dubitatur, fluxiones fluentium inventarum viciſſim colligendæ ſunt, & cum Fluxionibus ſub initio propoſitis comparandæ. Nam ſi prodeunt æquales, Concluſio recte ſe habet: ſin minus, corrigendæ ſunt fluentes ſic, ut earum Fluxiones fluxionibus ſub initio propoſitis æquentur. Nam & fluens pro lubitu aſſumi poteſt, & aſſumptio corrigi, ponendo fluxionem fluentis aſſumptæ æqualem Fluxioni propoſitæ, & terminos homologos inter ſe comparando.

Et his principiis via ad majora ſternitur.



ENUMERATIO
LINEARUM
TERTII ORDINIS.

ENUMERATIO
LINEARUM
TERTII ORDINIS.

I. *Linearum Ordines.*

Lineæ Geometricæ ſecundum numerum dimenſionum æquationis qua relatio inter Ordinatas & Abſciſſas definitur, vel (quod perinde eſt) ſecundum numerum punctorum in quibus a linea recta ſecari poſſunt, optime diſtinguuntur in Ordines. Qua ratione linea primi Ordinis erit Recta ſola, eæ ſecundi ſive quadratici Ordinis erunt ſectiones Conicæ & Circulus, & eæ tertii ſive cubici Ordinis Parabola Cubica, Parabola *Neiliana*, Ciſſois veterum, & reliquæ quas hic enumerare ſuſcepimus. Curva autem primi Generis, (ſiquidem recta inter Curvas non eſt numeranda) eadem eſt cum Linea ſecundi Ordinis, & Curva ſecundi Generis eadem cum Linea Ordinis tertii. Et Linea Ordinis infiniteſimi ea eſt quam recta in punctis infinitis ſecare poteſt, qualis eſt Spiralis, Cyclois, Quadratrix, & linea omnis quæ per radii vel rotæ revolutiones infinitas generatur.

II. *Proprietates Sectionum Conicarum competunt Curvis ſuperiorum Generum.*

Sectionum Conicarum proprietates præcipuæ a Geometris paſſim traduntur. Et conſimiles ſunt proprietates Curvarum ſecundi Generis, & reliquarum, ut ex ſequenti proprietatum præcipuarum enumeratione conſtabit.

1. *De Curvarum ſecundi generis Ordinatis, Diametris, Verticibus, Centris, Axibus.*

Si rectæ plures parallelæ & ad Conicam ſectionem utrinque terminatæ ducantur, recta duas earum biſecans biſecabit alias omnes, ideoque dicitur *Diameter* figuræ & rectæ biſectæ dicuntur *Ordinatim applicatæ* ad Diametrum, & concurſus omnium Diametrorum eſt *Centrum* figuræ, & interſectio Curvæ & diametri *Vertex* nominatur, & diameter illa *Axis* eſt cui Ordinatim applicatæ inſiſtunt ad angulos rectos. Et ad eundem modum in Curvis ſecundi Generis, ſi rectæ duæ quævis parallelæ ducantur occurrentes Curvæ in tribus punctis: recta quæ ita ſecat has parallelas ut ſumma duarum partium ex uno ſecantis latere ad Curvam terminatarum æquetur parti tertiæ ex altero latere ad curvam terminatæ, eodem modo ſecabit omnes alias his parallelas curvæque in tribus punctis occurrentes rectas, hoc eſt, ita ut ſumma partium duarum ex uno ipſius latere ſemper æquetur parti tertiæ ex altero latere. Has itaque tres partes quæ hinc inde æquantur, *Ordinatim applicatas*, & rectam ſecantem cui Ordinatim applicantur *Diametrum*, & interſectionem diametri & Curvæ *Verticem*, & concurſum duarum Diametrorum *Centrum* nominare licet. Diameter autem ad Ordinatas rectangula ſi modo aliqua ſit, etiam *Axis* dici poteſt, & ubi omnes Diametri in eodem puncto concurrunt, iſtud erit *Centrum generale*.

2. *De Aſymptotis & earum proprietatibus.*

Hyperbola primi generis duas *Aſymptotos*, ea ſecundi tres, ea tertii quatuor & non plures habere poteſt, & ſic in reliquis. Et quemadmodum partes lineæ cujuſvis rectæ inter Hyperbolam Conicam & duas ejus Aſymptotos ſunt hinc inde æquales: ſic in Hyperbolis ſecundi Generis ſi ducatur recta quævis ſecans tam Curvam quam tres ejus Aſymptotos in tribus punctis, ſumma duarum partium iſtius rectæ quæ a duobus quibuſvis Aſymptotis in eandem plagam ad duo puncta Curvæ extenduntur, æqualis erit parti tertiæ quæ a tertia Aſymptoto in plagam contrariam ad tertium Curvæ punctum extenditur.

3. *De Lateribus rectis & tranſverſis.*

Et quemadmodum in Conicis ſectionibus non Parabolicis quadratum Ordinatim applicatæ, hoc eſt, rectangulum Ordinatarum quæ ad contrarias partes Diametri ducuntur, eſt ad rectangulum partium Diametri quæ ad Vertices Ellipſeos vel Hyperbolæ terminantur, ut data quædam linea quæ dicitur *Latus rectum*, ad partem Diametri quæ inter Vertices jacet & dicitur *Latus tranſverſum* : ſic in Curvis non Parabolicis ſecundi Generis parallelepipedum ſub tribus Ordinatim applicatis eſt ad parallelepipedum ſub partibus Diametri ad Ordinatas & tres Vertices figuræ abſciſſis, in ratione quadam data : in qua ratione ſi ſumantur tres rectæ ad tres partes diametri inter vertices figuræ ſitas, ſingulæ ad ſingulas, tunc illæ tres rectæ dici poſſunt *Latera recta* figuræ, & illæ partes Diametri inter Vertices *Latera tranſverſa*. Et ſicut in Parabola Conica quæ ad unam & eandem diametrum unicum tantum habet Verticem, rectangulum ſub Ordinatis æquatur rectangulo ſub parte Diametri quæ ad Ordinatas & Verticem abſcinditur & recta quadam data quæ *Latus rectum* dicitur : ſic in Curvis ſecundi Generis quæ non niſi duos habent Vertices ad eandem Diametrum, parallelepipedum ſub Ordinatis tribus æquatur parallelepipedo ſub duabus partibus Diametri ad Ordinatas & Vertices illos duos abſciſſis & recta quadam data quæ proinde *Latus rectum* dici poteſt.

4. *De Ratione contentorum ſub Parallelarum ſegmentis.*

Denique ſicut in Conicis ſectionibus ubi duæ parallelæ ad Curvam utrinque terminatæ ſecantur a duabus parallelis ad Curvam utrinque terminatis, prima a tertia, & ſecunda a quarta, rectangulum partium primæ eſt ad rectangulum partium tertiæ ut rectangulum partium ſecundæ ad rectangulum partium quartæ : ſic ubi quatuor tales rectæ occurrunt Curvæ ſecundi Generis, ſingulæ in tribus punctis, parallelepipedum partium primæ rectæ erit ad parallelepipedum partium tertiæ, ut parallelepipedum partium ſecundæ ad parallelepipedum partium quartæ.

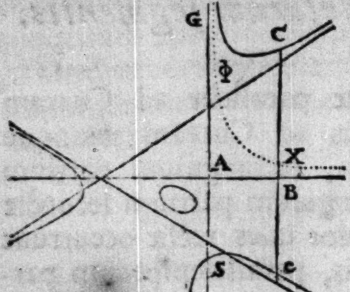
5. *De Cruribus Hyperbolicis & Parabolicis & eorum plagis.*

Curvarum ſecundi & ſuperiorum Generum æque atque primi crura omnia in infinitum progredientia vel *Hyperbolici* ſunt generis vel *Parabolici*. Crus *Hyperbolicum* voco quod ad Aſymptoton aliquam in infinitum appropinquat; *Parabolicam* quod Aſymptoto deſtituitur. Hæc crura ex Tangentibus optime dignoſcuntur. Nam ſi punctum contactus in infinitum abeat, Tangens cruris Hyperbolici cum Aſymptoto coincidet, & Tangens cruris Parabolici in infinitum recedet, evaneſcet & nullibi reperietur. Invenitur igitur Aſymptotos cruris cujuſvis quærendo Tangentem cruris illius ad punctum infinite diſtans. Plaga autem cruris infiniti invenitur quærendo poſitionem rectæ cujuſvis quæ Tangenti parallela eſt ubi punctum contactus in infinitum abit. Nam hæc recta in eandem plagam cum crure infinito dirigitur.

III. *Reductio Curvarum omnium Generis Secundi ad æquationum caſus quatuor.*

Lineæ omnes Ordinis primi, tertii, quinti, ſeptimi & imparis cujuſque duo habent ad minimum crura in infinitum verſus plagas oppoſitas progredientia. Et lineæ omnes tertii Ordinis duo habent ejuſmodi crura in plagas oppoſitas progredientia in quas nulla alia earum crura infinita (præterquam in Parabola *Carteſiana*) tendunt.

CAS. I.

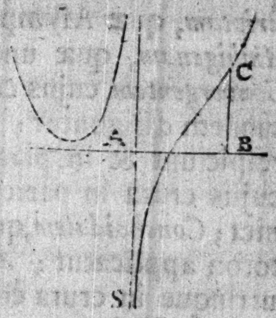


Si crura illa ſint Hyperbolici generis, ſit GAS eorum Aſymptotos, & huic parallela agatur recta quævis CBc ad Curvam utrinque (ſi fieri poteſt) terminata, eademque biſecetur in puncto X, & locus puncti illius X erit Hyperbola Conica (puta XΦ) cujus una Aſymptotos eſt AG. Sit ejus altera Aſymptotos AB, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſi AB dicatur x, & BC y, ſemper induet hanc formam $xyy + ey = ax^3 + bxx + cx + d$. Ubi termini, e, a, b, c, d, deſignant quantitates datas ſignis ſuis + & — affectas, quarum quælibet deeſſe poſſunt modo ex earum defectu figura in ſectionem Conicam non vertatur. Poteſt autem Hyperbola illa Conica cum Aſymptotis ſuis coincidere, id eſt punctum X in recta AB locari: & tunc terminus $+ ey$ deeſt.

CAS. II.

At ſi recta illa CBc non poteſt utrinque ad Curvam terminari, ſed Curvæ in unico tantum puncto occurrit: age quamvis poſitione datam rectam AB Aſymptoto AS occurrentem in A, ut & aliam quamvis BC Aſymptoto illi parallelam Curvæque occurrentem in puncto C, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

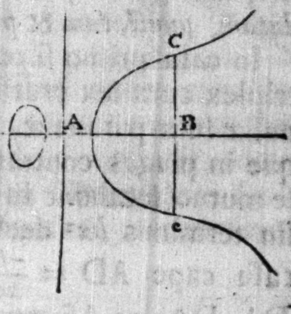
$$xy = ax^3 + bx^2 + cx + d.$$



CAS. III.

Quod ſi crura illa oppoſita Parabolici ſint generis, recta CBc ad Curvam utrinque, ſi fieri poteſt, terminata in plagam crurum ducatur & biſecetur in B, & locus puncti B erit Linea recta. Sit iſta AB, terminata ad datum quodvis punctum A, & æquatio qua relatio inter Ordinatam BC & Abſciſſam AB definitur, ſemper induet hanc formam,

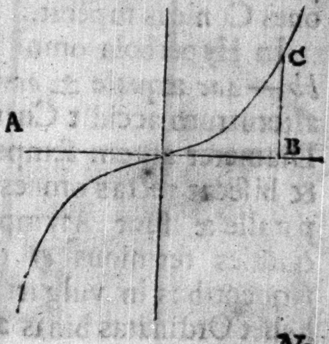
$$yy = ax^3 + bx^2 + cx + d.$$



CAS. IV.

At vero ſi recta illa CBc in unico tantum puncto occurrat Curvæ, ideoque ad Curvam utrinque terminari non poſſit: ſit punctum illud C, & incidat recta illa ad punctum B in rectam quamvis aliam poſitione datam & ad datum quodvis punctum A terminatam AB: & æquatio qua relatio inter Ordinatam BC & Abſciſſam AC definitur, ſemper induet hanc formam,

$$y = ax^3 + bx^2 + cx + d.$$



Nomina formarum.

Enumerando Curvas horum caſuum, Hyperbolam vocabimus, *Inſcriptam*, quæ tota jacet in Aſymptoton angulo ad inſtar Hyperbolæ conicæ; *Circumſcriptam*, quæ Aſymptotos ſecat & partes Abſciſſas in ſinu ſuo amplectitur; *Ambigenam*, quæ uno crure infinito inſcribitur & altero circumſcribitur; *Convergentem*, cujus crura concavitate ſua ſeinvicem reſpiciunt & in plagam eandem diriguntur; *Divergentem*, cujus crura convexitate ſua ſeinvicem reſpiciunt & in plagas contrarias diriguntur; *Cruribus Contrariis præditam*, cujus crura in partes contrarias convexa ſunt & in plagas contrarias infinita; *Conchoidalem*, quæ vertice concavo & cruribus divergentibus ad Aſymptoton applicatur; *Anguineam*, quæ flexibus contrariis Aſymptoton ſecat & utrinque in crura contraria producitur; *Cruciformem*, quæ conjugatam decuſſat; *Nodatam*, quæ ſeipſam decuſſat in orbem redeundo; *Cuſpidatam*, cujus partes duæ in angulo contactus concurrunt & ibi terminantur; *Punctatam*, quæ conjugatam habet Ovalem infinite parvam id eſt punctum; & *Puram*, quæ per impoſſibilitatem duarum radicum Ovali, Nodo, Cuſpide & Puncto conjugato privatur. Eodem ſenſu Parabolam quoque *convergentem*, *divergentem*, *cruribus contrariis præditam*, *cruciformem*, *nodatam*, *cuſpidatam*, *punctatam* & *puram* nominabimus.

In caſu primo ſi terminus ax^3 affirmativus eſt, Figura erit Hyperbola triplex cum ſex cruribus Hyperbolicis quæ juxta tres Aſymptotos quarum nullæ ſunt parallelæ, in infinitum progrediuntur, binæ juxta unamquamque in plagas contrarias. Et hæ Aſymptoti ſi terminus bx^2 non deeſt, ſe mutuo ſecabunt in tribus punctis triangulum (Ddδ) inter ſe continentes, ſin terminus bx^2 deeſt, convergent omnes ad idem punctum. In priori caſu cape $AD = \frac{-b}{2a}$, & $Ad = A\delta = \frac{b}{2\sqrt{a}}$, ac junge Dd, Dδ, & erunt Ad, Dd, Dδ tres Aſymptoti. In poſteriori duc Ordinatam quamvis BC, Ordinatæ principali AG parallelam, & in ea utrinque producta cape hinc inde BF & Bf ſibi mutuo æquales & in ea ratione ad AB quam habet $\sqrt{a}$ ad 1, jungeque AF, Af & erunt AG, AF, Af tres Aſymptoti. Hanc Hyperbolam vocamus *Redundantem*, quia numero crurum Hyperbolicorum Sectiones Conicas ſuperat.

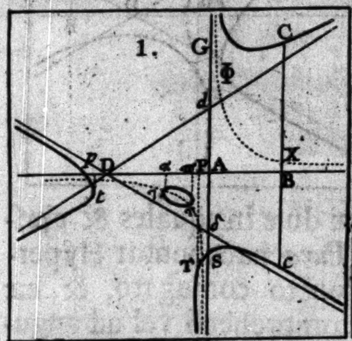
In Hyperbola omni Redundante, ſi neque terminus ey deſit, neque ſit $bb - 4ac$ æquale $\pm ae\sqrt{a}$, Curva nullam habebit Diametrum, ſin eorum alterutrum accidit Curva habebit unicam Diametrum, & tres ſi utrumque. Diameter autem ſemper tranſit per interſectionem duarum Aſymptoton, & biſecat rectas omnes quæ ad Aſymptotos illas utrinque terminantur & parallelæ ſunt Aſymptoto tertiæ. Eſtque Abſciſſa AB diameter Figuræ quoties terminus ey deeſt. *Diametrum* vero abſolute dictam hic & in ſequentibus in vulgari ſignificatu uſurpo, nempe pro Abſciſſa quæ paſſim habet Ordinatas binas æquales ad idem punctum hinc inde inſiſtentes.

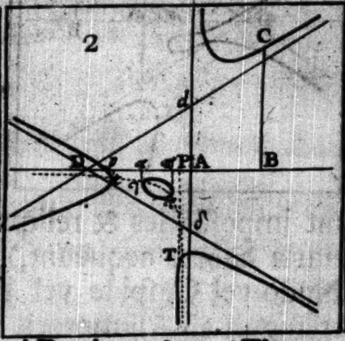
IV. *Enu-*

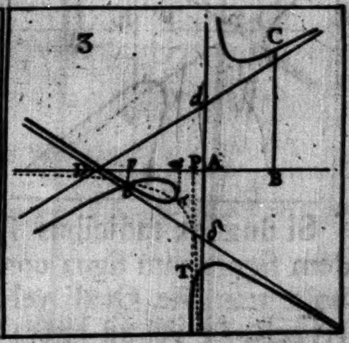
IV. *Enumeratio Curvarum.*

1. *De Hyperbolis novem redundantibus quæ diametro deſtituuntur & tres habent Aſymptotos triangulum capientes.*

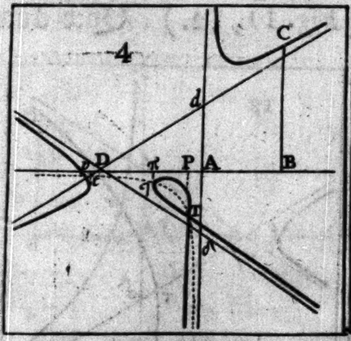
Si Hyperbola redundans nullam habet diametrum, quærantur Æquationis hujus $ax^4 + bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ radices quatuor ſeu valores ipſius x. Eæ ſunto AP, Aϖ, Aπ, Ap. Erigantur Ordinatæ PT, ϖτ, πϒ, pt, & hæ tangent Curvam in punctis totidem T, τ, ϒ, t, & tangendo dabunt limites Curvæ per quos Species ejus innoteſcet.

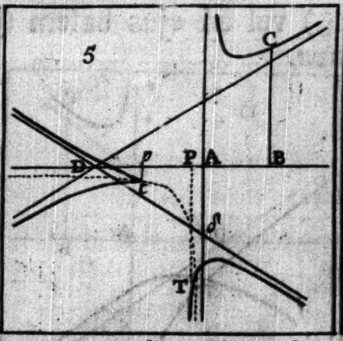


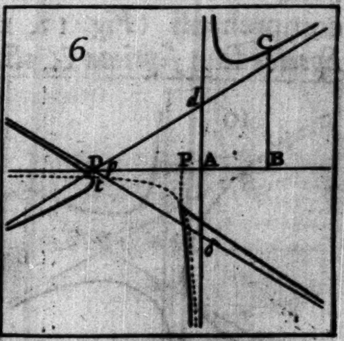




Nam ſi radices omnes AP, Aϖ, Ap, (*Fig.* 1, 2.) ſunt reales, ejuſdem ſigni & inæquales, Curva conſtat ex tribus Hyperbolis, (inſcripta, circumſcripta & ambigena) cum *Ovali*. Hyperbolarum una jacet verſus D, altera verſus d, tertia verſus δ, & Ovalis ſemper jacet intra Triangulum Ddδ, atque etiam inter medios limites ϒ & τ, in quibus utique tangitur ab Ordinatis ϖϒ & ϖτ. Et hæc eſt *Species prima*.



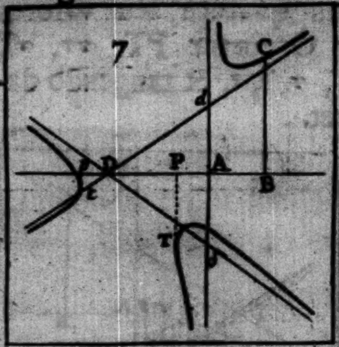


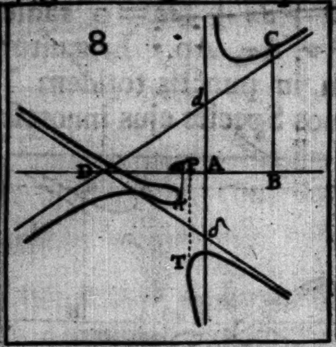


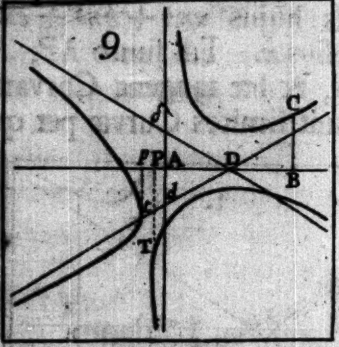
Si e radicibus duæ maximæ Aπ, Ap, (*Fig.* 3.) vel duæ minimæ AP, Aϖ (*Fig.* 4.) æquantur inter ſe, & ejuſdem ſunt ſigni cum alteris duobus, Ovalis & Hyperbola circumſcripta ſibi invicem junguntur coeuntibus earum punctis contactus ϒ & t vel T & τ, & crura Hyperbolæ ſeſe decuſſando in Ovalem continuantur, figuram *Nodatam* efficientia. Quæ *Species* eſt *ſecunda*.

Si e radicibus tres maximæ Ap, Aπ, Aϖ, (*Fig.* 5.) vel tres minimæ Aπ, Aϖ, AP (*Fig.* 6.) æquentur inter ſe, Nodus in *Cuſpidem* acutiſſimum convertetur. Nam crura duo Hyperbolæ circumſcriptæ ibi in angulo contactus concurrent & non ultra producentur. Et hæc eſt *Species tertia.*

Si e radicibus duæ mediæ Aϖ & Aπ (*Fig.* 7.) æquentur inter ſe, puncta contactus τ & ȷ coincidunt, & propterea Ovalis interjecta in punctum evanuit, & conſtat figura ex tribus Hyperbolis, inſcripta, circumſcripta & ambigena cum *Puncto* conjugato. Quæ eſt *Species quarta.*

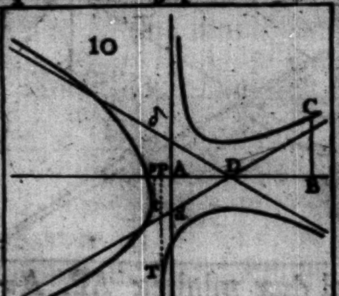


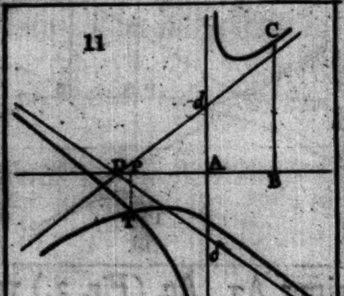


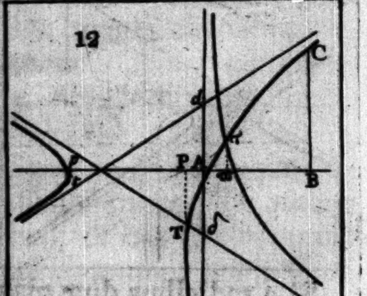


Si duæ ex radicibus ſunt impoſſibiles & reliquæ duæ inæquales & ejuſdem ſigni (nam ſigna contraria habere nequeunt,) *Puræ* habebuntur Hyperbolæ tres ſine Ovali vel Nodo vel Cuſpide vel Puncto conjugato, & hæ Hyperbolæ vel ad latera trianguli ab Aſymptotis comprehenſi vel ad angulos ejus jacebunt; & perinde *Speciem* vel *quintam* (*Fig.* 7. 8.) vel *ſextam* (*Fig.* 9, 10.) conſtituent.

Si e radicibus duæ ſunt æquales & alteræ duæ vel impoſſibiles ſunt (*Fig.* 11. 13.) vel reales (*Fig.* 12. 14.) cum ſignis quæ a ſignis æqualium radicum diverſa ſunt, figura *Cruciformis* habebitur, nempe duæ ex Hyperbolis ſeinvicem decuſſabunt idque vel ad verticem trianguli ab Aſymptotis comprehenſi (*Fig.* 13, 14.) vel ad ejus baſem (*Fig.* 11, 12.) Quæ duæ *Species* ſunt *ſeptima* & *octava.*

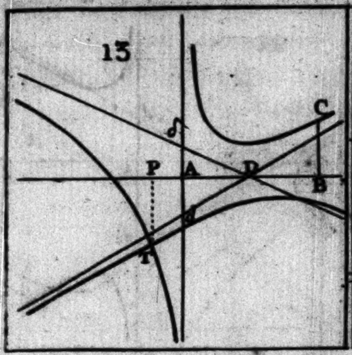


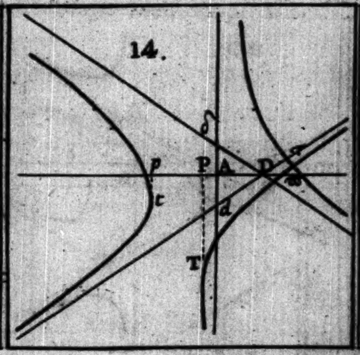


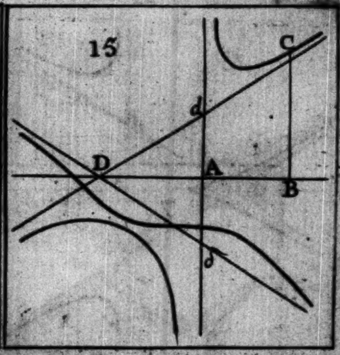


Si denique radices omnes ſunt impoſſibiles (*Fig.* 15.) vel ſi omnes ſunt reales & inæquales (*Fig.* 16.) & earum duæ ſunt affirmativæ & alteræ duæ negativæ, tunc duæ habebuntur Hyperbolæ ad angulos oppoſitos duarum Aſymptoton cum Hyperbola *Anguinea* circa Aſymptoton tertiam. Quæ *Species* eſt *nona.*

Et hi ſunt omnes radicum caſus poſſibiles. Nam ſi duæ radices ſunt æquales inter ſe, & aliæ duæ ſunt etiam inter ſe æquales, Figura evadet Sectio Conica cum Linea recta.



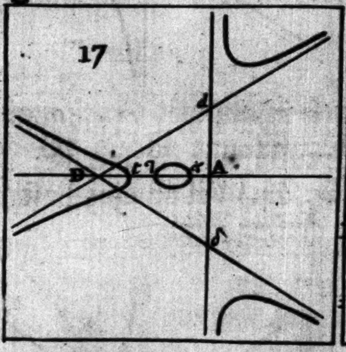


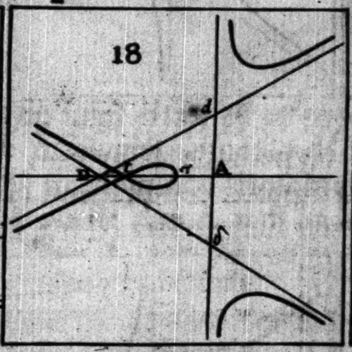


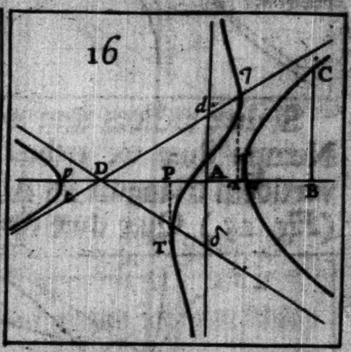
2. *De Hyperbolis duodecim redundantibus unicam tantum Diametrum habentibus.*

Si Hyperbola redundans habet unicam tantum Diametrum, ſit ejus Diameter Abſciſſa AB, & æquationis hujus $ax^3 + bx^2 + cx + d = 0$ quære tres radices ſeu valores x.

Si radices illæ ſunt omnes reales & ejuſdem ſigni, Figura conſtabit ex *Ovali* intra triangulum Ddδ (*Fig.* 17.) jacente & tribus Hyperbolis ad angulos ejus, nempe Circumſcripta ad angulum D & Inſcriptis duabus ad angulos d & δ. Et hæc eſt *Species decima.*







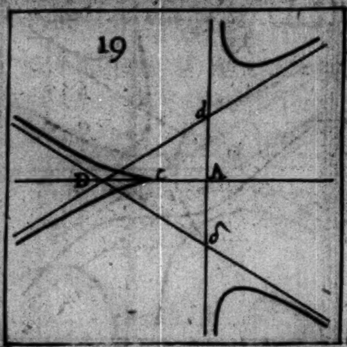
Si radices duæ majores ſunt æquales & tertia ejuſdem ſigni, crura Hyperbolæ jacentis verſus D (*Fig.* 18.) ſeſe decuſſabunt in forma *Nodi* propter contactum Ovalis. Quæ *Species* eſt *undecima.*

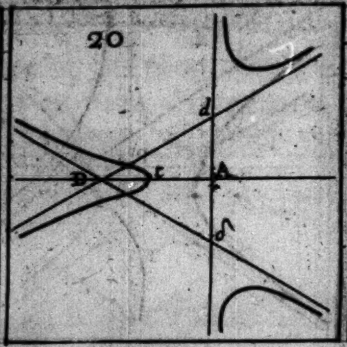
Si tres radices ſunt æquales, Hyperbola iſta fit *Cuſpidata* ſine Ovali, (*Fig.* 19.) Quæ *Species* eſt *duodecima.*

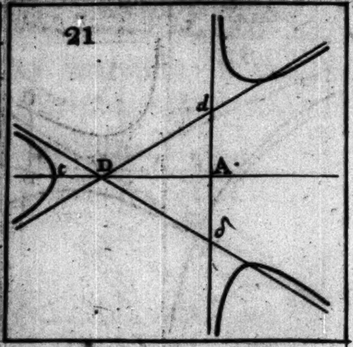
Si radices duæ minores ſunt æquales & tertia ejuſdem ſigni, Ovalis in *Punctum* evanuit, (*Fig.* 20.) Quæ *Species* eſt *decima tertia.*

In ſpeciebus quatuor noviſſimis Hyperbola quæ jacet verſus D, Aſymptotos in ſinu ſuo amplectitur, reliquæ duæ in ſinu Aſymptoton jacent.

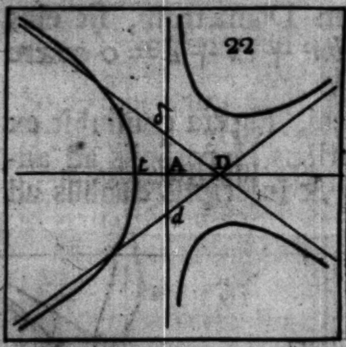
Si duæ ex radicibus ſunt impoſſibiles habebuntur tres Hyperbolæ *Puræ* ſine Ovali decuſſatione vel cuſpide. Et hujus caſus *Species* ſunt quatuor: nempe *decima quarta* ſi Hyperbola Circumſcripta jacet verſus D, (*Fig.* 20.)

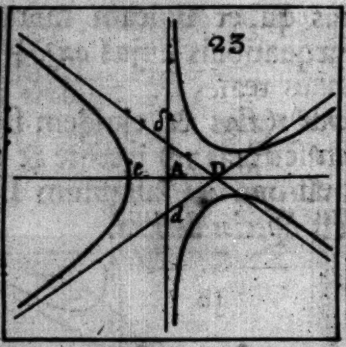


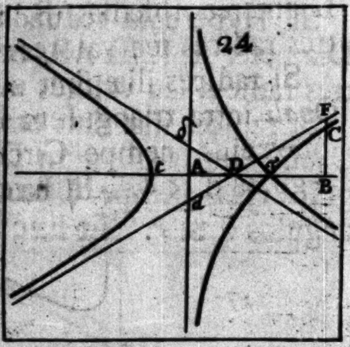




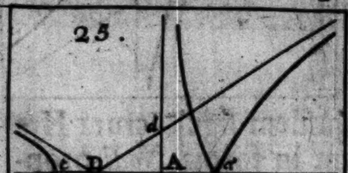
& *decima quinta* ſi Hyperbola Inſcripta jacet verſus D, (*Fig.* 21.) *decima ſexta* ſi Hyperbola Circumſcripta jacet ſub baſi dδ trianguli Ddδ, (*Fig.* 22.) & *decima ſeptima* (*Fig.* 23.) ſi Hyperbola inſcripta jacet ſub eadem baſi.

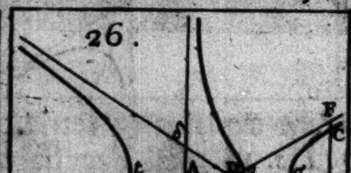


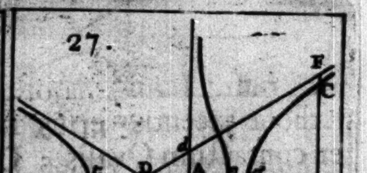




Si duæ radices ſunt æquales & tertia ſigni diverſi figura erit *Cruciformis*. Nempe duæ ex tribus Hyperbolis ſeinvicem decuſſabunt idque vel ad verticem trianguli ab Aſymptotis comprehenſi (*Fig.* 24.) vel ad ejus baſem, (*Fig.* 25.) Quæ duæ *Species* ſunt *decima octava*, & *decima nona*.

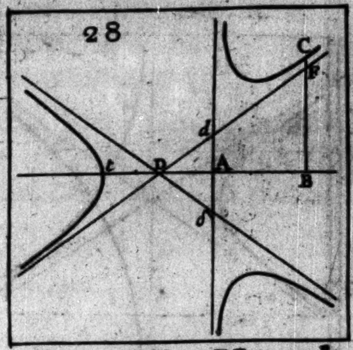


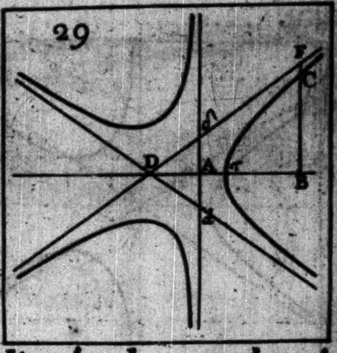


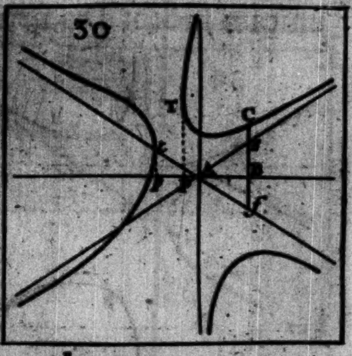


Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni diverſi, duæ habebuntur Hyperbolæ in oppoſitis angulis duarum Aſymptoton cum *Con-*

Conchoïdali intermedia. *Conchoidalis* autem vel jacebit ad eaſdem partes Aſymptoti ſuæ cum triangulo ab Aſymptotis conſtituto, (*Fig.* 26.) vel ad partes contrarias, (*Fig.* 27.) Et hi duo caſus conſtituunt *Speciem vigeſimam* & *vigeſimam primam.*

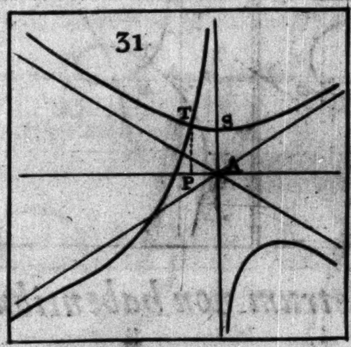


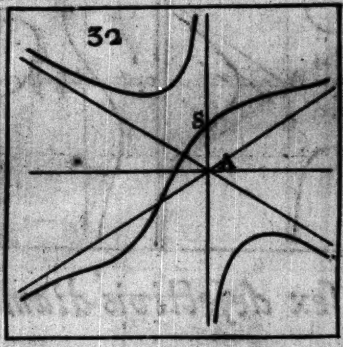


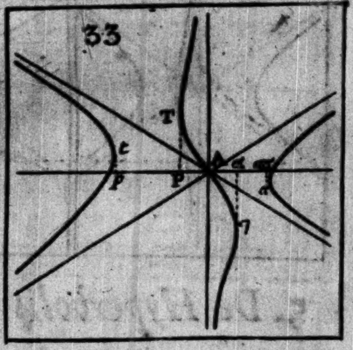


3. *De Hyperbolis duabus redundantibus cum tribus Diametris.*

Hyperbola redundans quæ habet tres diametros, conſtat ex tribus Hyperbolis in ſinubus Aſymptoton jacentibus, idque vel ad angulos trianguli ab Aſymptotis comprehenſi (*Fig.* 28.) vel ad ejus latera (*Fig.* 29.) Caſus prior dat *Speciem vigeſimam ſecundam*, & poſterior *Speciem vigeſimam tertiam.*



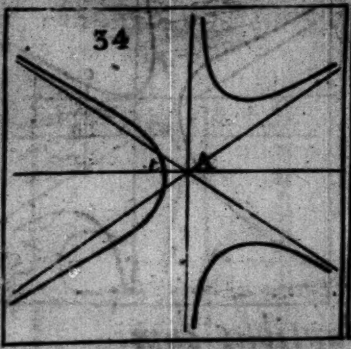




4. *De Hyperbolis novem redundantibus cum Aſymptotis tribus ad commune punctum convergentibus.*

Si tres Aſymptoti in puncto communi ſe mutuo decuſſant, vertuntur ſpecies quinta & ſexta in *vigeſimam quartam*, (*Fig.* 30.) ſeptima & octava in *vigeſimam quintam*, (*Fig.* 31.) & nona in *vigeſimam ſextam* (*Fig.* 32.) ubi Anguinea non tranſit per concurſum Aſymptoton, & in *vigeſimam ſeptimam* ubi tranſit per concurſum illum, (*Fig.* 33.) quo caſu termini *b* ac *d* deſunt, & concurſus Aſymptoton eſt Centrum figuræ ab omnibus ejus partibus oppoſitis æqualiter diſtans. Et hæ quatuor ſpecies diametrum non habent.

Vertuntur etiam *Species* decima quarta ac decima ſexta in *vigeſimam octavam*, (*Fig.* 34.) decima quinta ac decima ſeptima in *vigeſimam nonam*, (*Fig.* 35.) decima octava & decima nona in *triceſimam*, (*Fig.* 36) & vigeſima cum vigeſima prima in *triceſimam primam*, (*Fig.* 37.) Et hæ ſpecies unicam habent Diametrum.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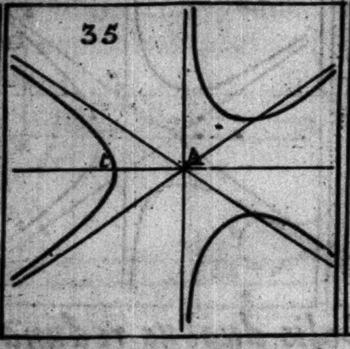
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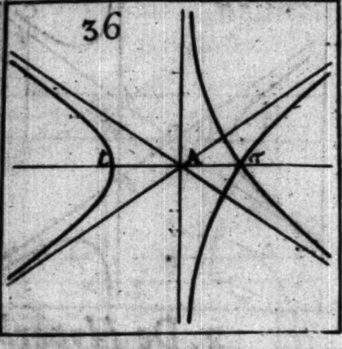
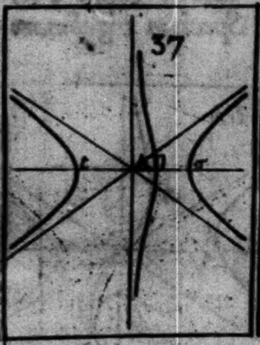

36

Ac denique ſpecies vigeſima ſecunda & vigeſima tertia vertuntur in *Speciem triceſimam ſecundam* cujus tres ſunt Diametri per concurſum Aſymptoton tranſeuntes. (*Fig.* 38.)

Quæ omnes converſiones facillime intelliguntur faciendo ut triangulum ab Aſymptotis comprehenſum diminuatur donec in punctum evaneſcat.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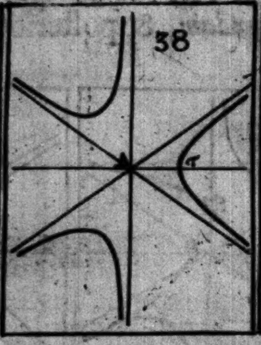
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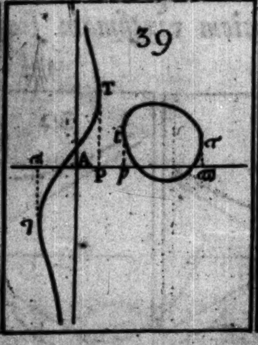
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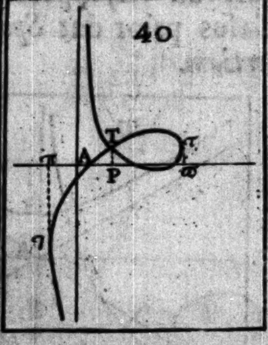
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5. *De Hyperbolis ſex defectivis diametrum non habentibus.*

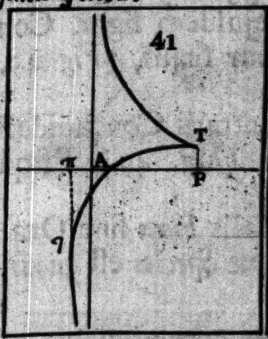
Si in primo æquationum caſu terminus ax^3 negativus eſt, Figura erit Hyperbola defectiva unicam habens Aſymptoton & duo tantum crura Hyperbolica juxta Aſymptoton illam in plagas contrarias infinite progredientia. Et Aſymptotos illa eſt Ordinata prima & principalis AG. Si terminus ey non deeſt figura nullam habebit Diametrum, ſi deeſt habebit unicam. In priori caſu ſpecies ſic enumerantur.

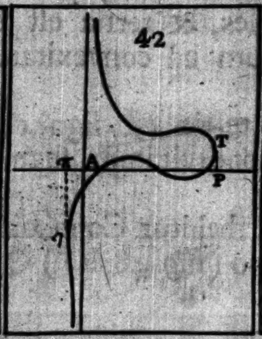
Si æquationis hujus $ax^4 = bx^3 + cx^2 + dx + \frac{1}{4}ee$ radices omnes Aπ, AP, Ap, Aϖ, (*Fig.* 39.) ſunt reales & inæquales, Figura erit Hyperbola Anguinea aſymptoton flexu contrario amplexa, cum *Ovali* conjugata. Quæ *Species* eſt *triceſima tertia*.

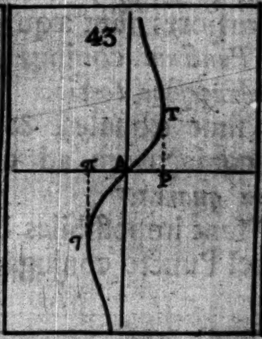
Si radices duæ mediæ AP & A*p* (*Fig.* 40) æquentur inter ſe, Ovalis & Anguinea junguntur ſeſe decuſſantes in forma *Nodi*. Quæ eſt *Species triceſima quarta*.

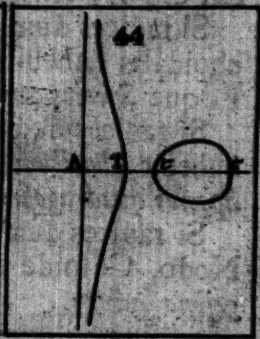
Si tres radices ſunt æquales, Nodus vertetur in *Cuſpidem* acutiſſimum in vertice Anguineæ, (*Fig.* 41) Et hæc eſt *Species triceſima quinta*.

Si e tribus radicibus ejuſdem ſigni duæ maximæ A*p* & Aϖ (*Fig.* 43) ſibi mutuo æquantur, Ovalis in *Punctum* evanuit. Quæ *Species* eſt *triceſima ſexta*.

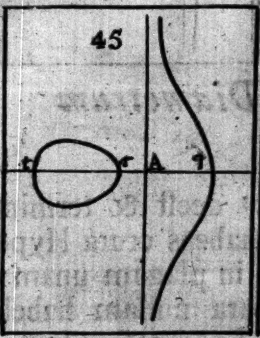


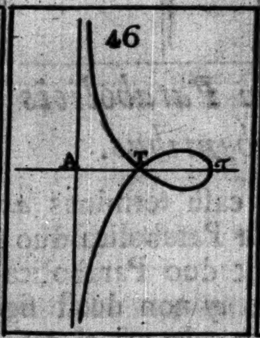


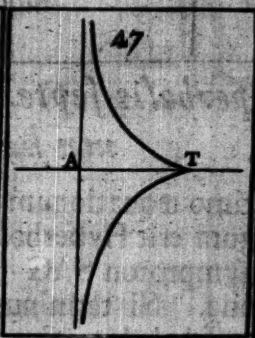


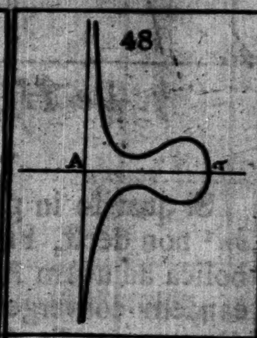


Si radices duæ quævis imaginariæ ſunt, ſola manebit Anguinea *Pura* ſine Ovali, Decuſſatione, Cuſpide vel Puncto conjugato. Si Anguinea illa non tranſit per punctum A (*Fig.* 42) *Species* eſt *triceſima ſeptima*; ſin tranſit per punctum illud A (id quod contingit ubi termini b ac d deſunt,) punctum illud A erit centrum figuræ rectas omnes per ipſum ductas & ad Curvam utrinque terminatas biſecans, (*Fig.* 43.) Et hæc eſt *Species triceſima octava*.









6. *De Hyperbolis ſeptem defectivis diametrum habentibus.*

In altero caſu ubi terminus ey deeſt & propterea figura Diametrum habet, ſi æquationis hujus $ax^3 = bx^2 + cx + d$ radices omnes AT, At, Aτ, (*Fig.* 44) ſunt reales, inæquales & ejuſdem ſigni, figura erit Hyperbola Conchoidalis cum *Ovali* ad convexitatem. Quæ eſt *Species triceſima nona*.

Si duæ radices ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni contrarii, *Ovalis* jacebit ad concavitatem Conchoidalis, *Fig.* 45.) Eſtque *Species quadrageſima.*

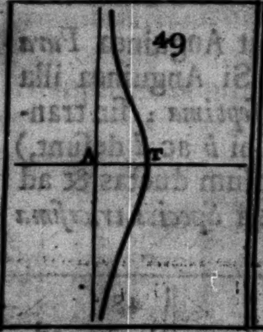
Si radices duæ minores AT, At, (*Fig.* 46) ſunt æquales, & tertia $A\tau$ eſt ejuſdem ſigni, Ovalis & Conchoidalis jungentur ſeſe decuſſando in modum *Nodi.* Quæ *Species* eſt *quadrageſima prima.*

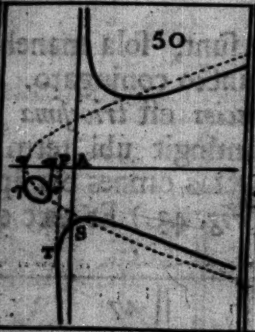
Si tres radices ſunt æquales, Nodus mutabitur in *Cuſpidem*, & figura erit *Ciſſois Veterum*, (*Fig.* 47.) Et hæc eſt *Species quadrageſima ſecunda.*

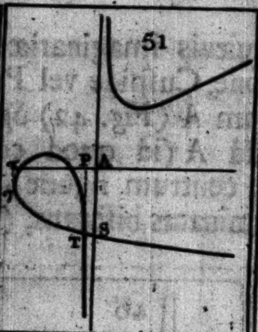
Si radices duæ majores ſunt æquales, & tertia eſt ejuſdem ſigni, Conchoidalis habebit *Punctum* conjugatum ad convexitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima tertia.*

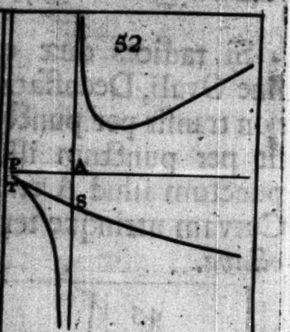
Si radices duæ ſunt æquales, & tertia eſt ſigni contrarii Conchoidalis habebit *Punctum* conjugatum ad concavitatem ſuam, (*Fig.* 49.) Eſtque *Species quadrageſima quarta.*

Si radices duæ ſunt impoſſibiles habebitur Conchoidalis *Pura* ſine Ovali, Nodo, Cuſpide vel Puncto conjugato (*Fig.* 48. 49.) Quæ *Species* eſt *quadrageſima quinta.*









7. *De Hyperbolis ſeptem Parabolicis Diametrum non habentibus.*

Si quando in primo æquationum caſu terminus ax^3 deeſt & terminus bx^2 non deeſt, Figura erit Hyperbola Parabolica duo habens crura Hyperbolica ad unam Aſymptoton SAG & duo Parabolica in plagam unam & eandem convergentia. Si terminus ey non deeſt figura nullam habebit diametrum, ſin deeſt habebit unicam. In priori caſu Species ſunt hæ.

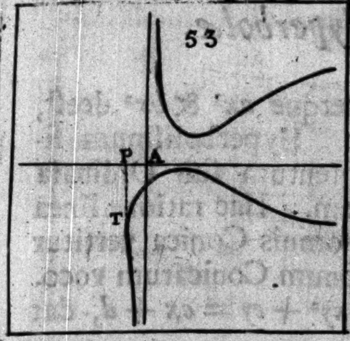
Si tres radices AP, $A\pi$, $A\tau$ (*Fig.* 50.) æquationis hujus $bx^3 + cx^2 + dx + \frac{1}{4}ee = 0$ ſunt inæquales & ejuſdem ſigni, figura conſtabit ex *Ovali* & aliis duabus Curvis quæ partim Hyperbolicæ ſunt & partim Parabolicæ. Nempe crura Parabolica continuo ductu junguntur cruribus Hyperbolicis ſibi proximis. Et hæc eſt *Species quadrageſima ſexta.*

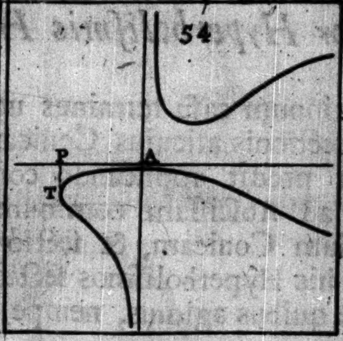
Si radices duæ minores ſunt æquales, & tertia eſt ejuſdem ſigni, Ovalis & una Curvarum illarum Hyperbolo-Parabolicarum junguntur & ſe decuſſant in formam *Nodi* (*Fig.* 51.) Quæ *Species* eſt *quadrageſima ſeptima.*

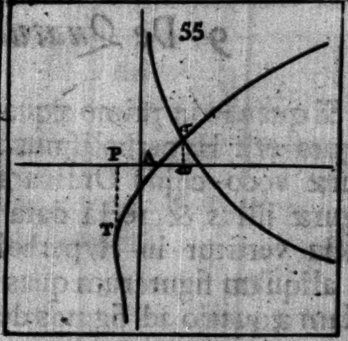
Si tres radices ſunt æquales, Nodus ille in Cuſpidem vertitur (*Fig.* 52.) Eſtque *Species quadrageſima octava.*

Si radices duæ majores ſunt æquales & tertia eſt ejuſdem ſigni, Ovalis in *Punctum* conjugatum evanuit (*Fig.* 53.) Quæ *Species* eſt *quadrageſima nona.*

Si duæ radices ſunt impoſſibiles, manebunt *Puræ* illæ duæ curvæ Hyperbolo-parabolicæ ſine Ovali, Decuſſatione, Cuſpide vel Puncto conjugato; & *Speciem quinquageſimam* conſtituent. (*Fig.* 53. 54.)

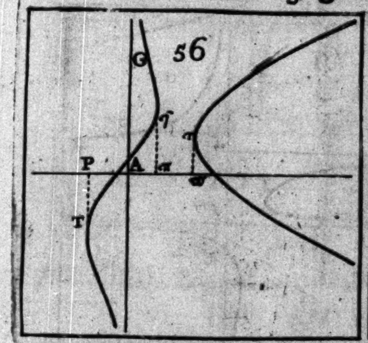


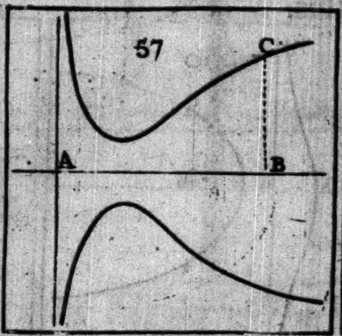


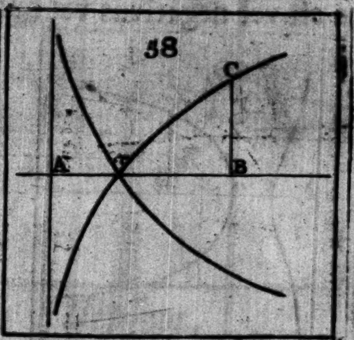


Si radices duæ ſunt æquales & tertia eſt ſigni contrarii, Curvæ illæ Hyperbolo-parabolicæ junguntur ſeſe decuſſando in morem crucis (*Fig.* 55.) Eſtque *Species quinquageſima prima.*

Si radices duæ ſunt inæquales & ejuſdem ſigni & tertia eſt ſigni contrarii, figura evadet Hyperbola Anguinea circa Aſymptoton AG, (*Fig.* 56) cum Parabola conjugata. Et hæc eſt *Species quinquageſima ſecunda.*







8. *De Hyperbolis quatuor Parabolicis Diametrum habentibus.*

In altero caſu ubi terminus *ey* deeſt & figura Diametrum habet, ſi duæ radices æquationis hujus $bx^2 + cx + d = 0$ ſunt impoſſibiles, duæ habentur figuræ Hyperbolo-parabolicæ a Diametro AB (*Fig.* 57.) hinc inde æqualiter diſtantes. Quæ *Species* eſt *quinquageſima tertia.*

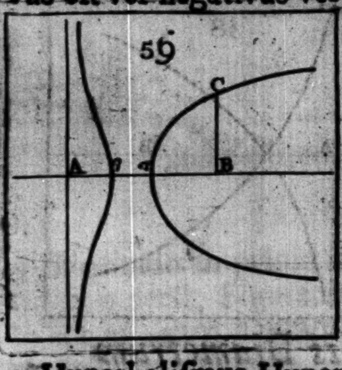
Si æquationis illius radices duæ ſunt impoſſibiles, Figuræ Hyperbolo-parabolicæ junguntur ſeſe decuſſantes in morem crucis; & *Speciem quinquageſimam quartam* conſtituunt. *Fig.* 58.

Si radices illæ ſunt inæquales & ejuſdem ſigni, habetur Hyperbola Conchoidalis cum Parabola ex eodem latere Aſymptoti (*Fig.* 59.) Eſtque *Species quinquageſima quinta.*

Si radices illæ ſunt ſigni contrarii, habetur Conchoidalis cum Parabola ad alteras partes Aſymptoti (*Fig.* 60.) Quæ *Species* eſt *quinquageſima ſexta.*

9. *De Quatuor Hyperboliſmis Hyperbolæ.*

Si quando in primo æquationum caſu terminus uterque ax^3 & bx^2 deeſt, figura erit Hyperboliſmus ſectionis alicujus Conicæ. Hyperboliſmum figuræ voco cujus Ordinata prodit applicando contentum ſub Ordinata figuræ illius & recta data ad Abſciſſam communem. Hac ratione linea recta vertitur in Hyperbolam Conicam, & ſectio omnis Conica vertitur in aliquam figurarum quas hic Hyperboliſmos ſectionum Conicarum voco. Nam æquatio ad figuras de quibus agimus, nempe $xy^2 + ey = cx + d$, dat $y = \frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2x}$ quæ generatur applicando contentum ſub Ordinata ſectionis Conicæ $\frac{e \pm \sqrt{ee + 4dx + 4cxx}}{2m}$ & recta data m, ad curvarum Abſciſſam communem x. Unde liquet quod figura genita Hyperboliſmus erit Hyperbolæ, Ellipſeos vel Parabolæ, perinde ut terminus cx affirmativus eſt vel negativus vel nullus.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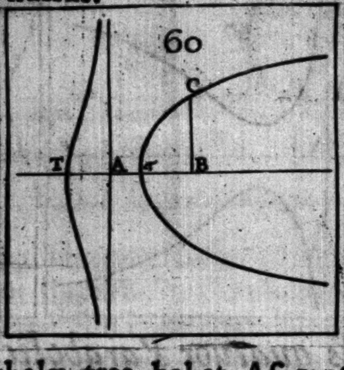
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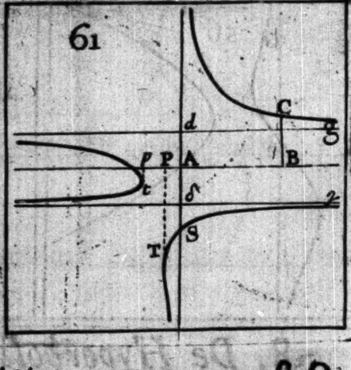
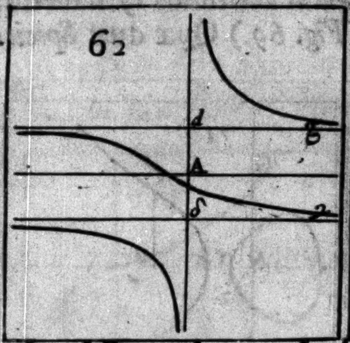

61

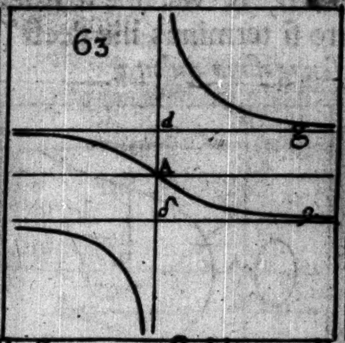
Hyperboliſmus Hyperbolæ tres habet Aſymptotos, quarum una eſt Ordinata prima & principalis Ad, alteræ duæ ſunt parallelæ Abſciſſæ AB, ab eadem hinc inde æqualiter diſtant. In Ordinata principali Ad, cape Ad, Aδ hinc inde æquales quantitati $\sqrt{c}$; & per puncta d ac δ age dg, $\delta\gamma$ Aſymptotos Abſciſſæ AB parallelas.

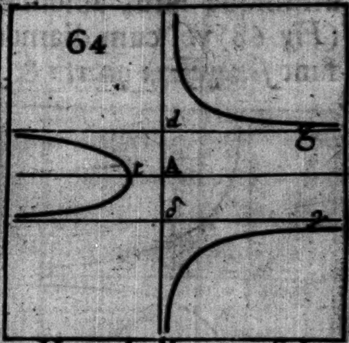
Ubi terminus ey non deeſt figura nullam habet diametrum. In hoc caſu, ſi æquationis hujus $cx^2 + dx + \frac{1}{4}ee = 0$ radices duæ AP, Ap (*Fig.* 61) ſunt reales & inæquales (nam æquales eſſe nequeunt niſi figura ſit Conica ſectio) figura conſtabit ex tribus Hyperbolis ſibi oppoſitis quarum una jacet inter

Aſymptotos parallelas & alteræ duæ jacent extra. Et hæc eſt *Species quinquageſima ſeptima.*

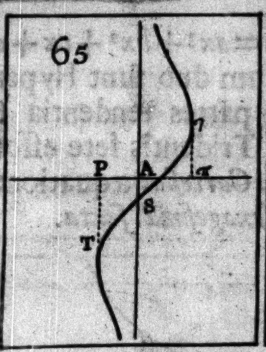
Si radices illæ duæ ſunt impoſſibiles, habentur Hyperbolæ duæ oppoſitæ extra Aſymptotos parallelas & Anguinea Hyperbolica intra eaſdem. Hæc figura duarum eſt ſpecierum. Nam centrum non habet ubi terminus *d* non deeſt (*Fig.* 62;) ſed ſi terminus ille deeſt punctum A eſt ejus centrum (*Fig.* 63.) Prior *Species* eſt *quinquageſima octava*, poſterior *quinquageſima nona.*

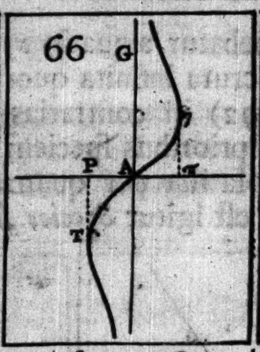


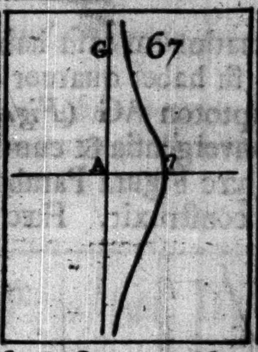


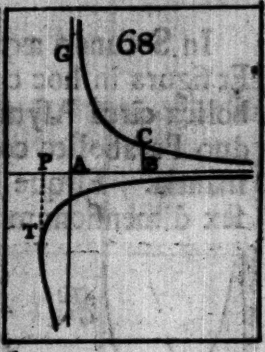


Quod ſi terminus *ey* deeſt, figura conſtabit ex tribus Hyperbolis oppoſitis, quarum una jacet inter Aſymptotos parallelas & alteræ duæ jacent extra ut in ſpecie quinquageſima quarta, & præterea diametrum habet quæ eſt Abſciſſa AB (*Fig.* 64.) Et hæc eſt *Species ſexageſima.*









10. *De tribus Hyperboliſmis Ellipſeos.*

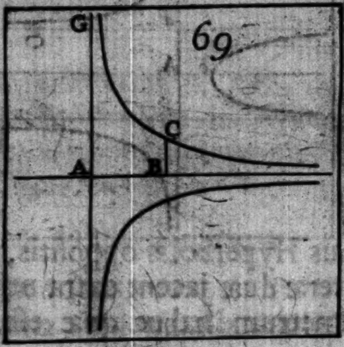
Hyperboliſmus Ellipſeos per hanc æquationem definitur $xy^2+ey=cx+d$, & unicam habet Aſymptoton quæ eſt Ordinata principalis Ad (*Fig.* 65.) Si terminus *ey* non deeſt, figura eſt Hyperbola Anguinea ſine diametro, atque etiam ſine centro ſi terminus *d* non deeſt. Quæ *Species* eſt *ſexageſima prima.*

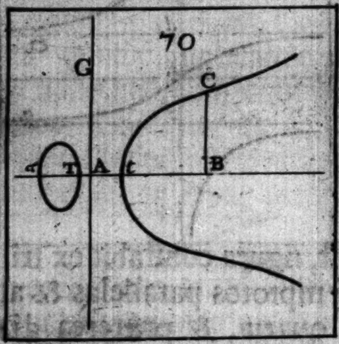
At ſi terminus *d* deeſt, figura habet centrum ſine diametro, & centrum ejus eſt punctum A (*Fig.* 66.) *Species* vero eſt *ſexageſima ſecunda.*

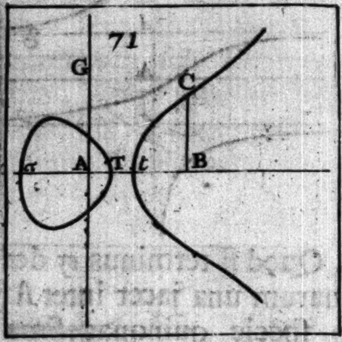
Et ſi terminus *ey* deeſt, & terminus *d* non deeſt, figura eſt Conchoidalis ad Aſymptoton AG *Fig.* 67,) habetque diametrum ſine centro, & diameter ejus eſt Abſciſſa AB. Quæ *Species* eſt *ſexageſima tertia.*

11. *De duobus Hyperboliſmis Parabolæ.*

Hyperboliſmus Parabolæ per hanc æquationem definitur $xy^2 + ey = d$; & duas habet Aſymptotos, Abſciſſam AB & Ordinatam primam & principalem AG. Hyperbolæ vero in hac figura ſunt duæ, non in Aſymptoton angulis oppoſitis ſed in angulis qui ſunt deinceps jacentes, idque ad utrumque latus abſciſſæ AB, & vel ſine diametro ſi terminus *ey* habetur, (*Fig.* 68) vel cum diametro ſi terminus ille deeſt (*Fig.* 69.) Quæ duæ *Species* ſunt *ſexageſima quarta* & *ſexageſima quinta*.

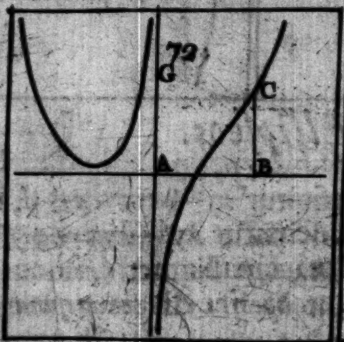


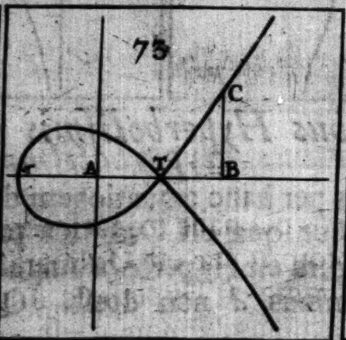


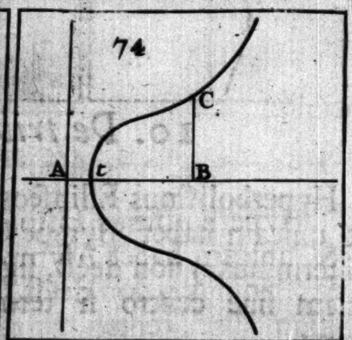


12. *De Tridente.*

In Secundo æquationum caſu habebatur æquatio $xy = ax^3 + bx^2 + cx + d$. Et figura in hoc caſu habet quatuor crura infinita quorum duo ſunt Hyperbolica circa Aſymptoton AG (*Fig.* 72) in contrarias partes tendentia & duo Parabolica convergentia & cum prioribus ſpeciem Tridentis fere efformantia. Eſtque hæc Figura Parabola illa per quam *Carteſius* æquationes ſex dimenſionum conſtruxit. Hæc eſt igitur *Species ſexageſima ſexta*.







13. *De Parabolis quinque divergentibus.*

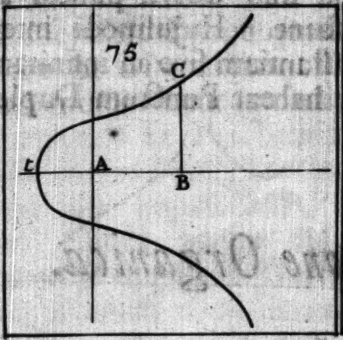
In Tertio caſu æquatio erat $y^2 = ax^3 + bx^2 + cx + d$, & Parabolam deſignat cujus crura divergunt ab invicem & in contrarias partes infinite progrediuntur. Abſciſſa AB eſt ejus diameter & Species ejus ſunt quinque ſequentes.

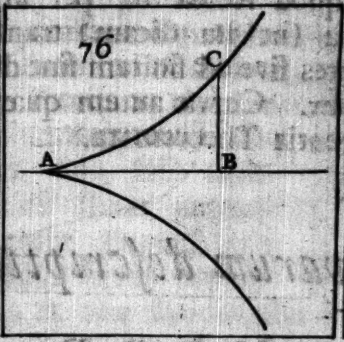
Si æquationis $ax^3+bx^2+cx+d=0$, radices omnes Aτ, AT, At ſunt reales & inæquales, figura eſt Parabola divergens Campaniformis cum *Ovali* ad verticem (*Fig.* 70, 71.) Et *Species* eſt *ſexageſima ſeptima.*

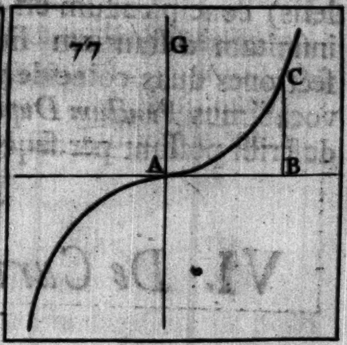
Si radices duæ ſunt æquales, Parabola prodit vel *Nodata* contingendo Ovalem (*Fig.* 73,) vel *Punctata* ob Ovalem infinite parvam (*Fig.* 74.) Quæ duæ *Species* ſunt *ſexageſima octava* & *ſexageſima nona.*

Si tres radices ſunt æquales Parabola erit *Cuſpidata* in vertice (*Fig.* 76.) Et hæc eſt Parabola *Neiliana* quæ vulgo *Semicubica* dicitur. Et eſt *Species ſeptuageſima.*

Si radices duæ ſunt impoſſibiles, habetur Parabola *Pura* campaniformis (*Fig.* 74, 75,) *Speciem ſeptuageſimam primam* conſtituens.







14. *De Parabola Cubica.*

In Quarto caſu æquatio erat $y=ax^3+bx^2+cx+d$, & hæc æquatio Parabolam deſignat quæ crura habet contraria & *Cubica* dici ſolet (*Fig.* 77.) Et ſic *Species* omnino ſunt *ſeptuaginta duæ.*

V. *Geneſis Curvarum per Umbras.*

Si in planum infinitum a puncto lucido illuminatum umbræ figurarum projiciantur, umbræ Sectionum Conicarum ſemper erunt Sectiones Conicæ, eæ Curvarum ſecundi Generis ſemper erunt Curvæ ſecundi Generis, eæ Curvarum tertii Generis ſemper erunt Curvæ tertii Generis, & ſic deinceps in infinitum.

Et quemadmodum Circulus umbram projiciendo generat Sectiones omnes Conicas, ſic Parabolæ quinque divergentes umbris ſuis generant & exhibent alias omnes ſecundi Generis Curvas; & ſic Curvæ quædam ſimpliciores aliorum Generum inveniri poſſunt quæ alias omnes eorundem Generum Curvas umbris ſuis a puncto lucido in planum projectis formabunt.

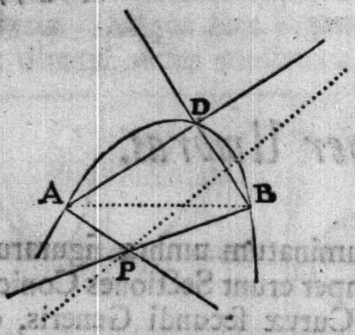
De Curvarum Punctis duplicibus.

Diximus Curvas ſecundi Generis a Linea recta in punctis tribus ſecari poſſe. Horum duo nonnunquam coincidunt. Ut cum Recta per Ovalem infinite parvam tranſit vel per concurſum duarum partium Curvæ ſe mutuo ſecantium vel in cuſpidem coeuntium ducitur. Et ſiquando Rectæ omnes in plagam cruris alicujus infiniti tendentes Curvam in unico tantum puncto ſecant, (ut fit in ordinatis Parabolæ *Carteſianæ* & Parabolæ cubicæ, nec non in rectis Abſciſſæ Hyperboliſmorum Hyperbolæ & Parabolæ parallelis) concipiendum eſt quod Rectæ illæ per alia duo Curvæ puncta ad infinitam diſtantiam ſita (ut ita dicam) tranſeunt. Hujuſmodi interſectiones duas coincidentes ſive ad finitam ſint diſtantiam ſive ad infinitam, vocabimus *Punctum Duplex*. Curvæ autem quæ habent Punctum Duplex deſcribi poſſunt per ſequentia Theoremata.

VI. *De Curvarum deſcriptione Organica.*

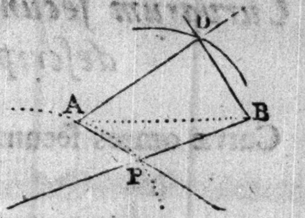
THEOR. I.

* Si Anguli duo magnitudine dati PAD, PBD circa polos poſitione datos A, B rotentur, & eorum crura AP, BP concurſu ſuo P percurrant Lineam rectam; crura duo reliqua AD, BD concurſu ſuo D deſcribent Sectionem Conicam per polos A, B tranſeuntem: præterquam ubi Linea illa recta tranſit per polorum alterutrum A vel B, vel anguli BAD, ABD ſimul evaneſcunt, quibus in caſibus punctum D deſcribet Lineam rectam.



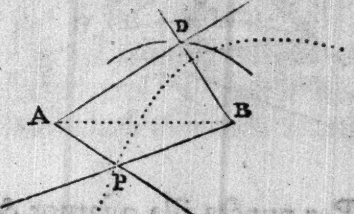
THEOR. II.

Si crura prima AP, BP concurſu ſuo P percurrant Sectionem Conicam per polum alterutrum A tranſeuntem, crura duo reliqua AD, BD concurſu ſuo D deſcribent Curvam ſecundi Generis per polum alterum B tranſeuntem & Punctum duplex habentem in polo primo A, per quem ſectio Conica tranſit: præterquam ubi anguli BAD, ABD ſimul evaneſcunt, quo caſu punctum D deſcribet aliam ſectionem Conicam per polum A tranſeuntem.



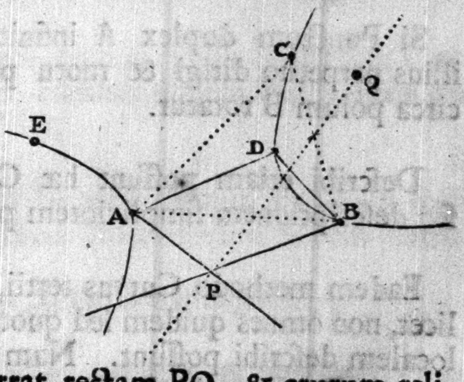
THEOR. III.

At ſi ſectio Conica quam punctum P percurrit tranſeat per neutrum polorum A, B, punctum D deſcribet Curvam ſecundi vel tertii generis Punctum duplex habentem. Et Punctum illud duplex in concurſu crurum deſcribentium, AD, BD invenietur ubi anguli BAP, ABP ſimul evaneſcunt. Curva autem deſcripta ſecundi erit Generis ſi anguli BAD, ABD ſimul evaneſcunt, alias erit tertii Generis & alia duo habebit Puncta duplicia in polis A & B.

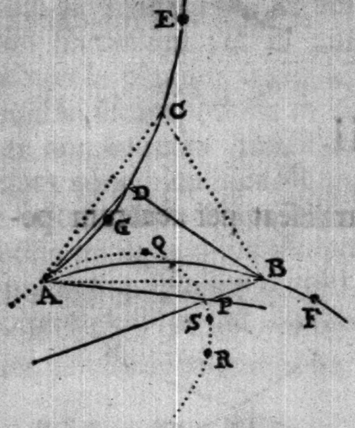


Sectionum Conicarum deſcriptio per data quinque puncta.

Jam ſectio Conica determinatur ex datis ejus punctis quinque & per eadem ſic deſcribi poteſt. Dentur ejus puncta quinque A, B, C, D, E. Jungantur eorum tria quævis A, B, C, & trianguli ABC rotentur anguli duo quivis CAB, CBA circa vertices ſuos A & B, & ubi crurum AC, BC interſectio C ſucceſſive applicatur ad puncta duo reliqua D, E, incidat interſectio crurum reliquorum AB & BA in puncta P & Q. Agatur & infinite producatur recta PQ, & anguli mobiles ita rotentur ut interſectio crurum AB, BA percurrat rectam PQ, & crurum reliquorum interſectio C deſcribet propoſitam ſectionem Conicam per *Theorema* primum.



Curvarum ſecundi generis Punctum duplex habentium deſcriptio per data ſeptem puncta.



Curvæ omnes ſecundi generis Punctum Duplex habentes determinantur ex datis earum punctis ſeptem quorum unum eſt Punctum illud duplex, & per eadem puncta ſic deſcribi poſſunt. Dentur Curvæ deſcribendæ puncta quælibet ſeptem A, B, C, D, E, F, G, quorum A eſt Punctum Duplex. Jungantur punctum A & alia duo quævis e punctis puta B & C; & trianguli ABC rotetur tum angulus CAB circa verticem ſuum A, tum angulorum reliquorum alteruter ABC circa verticem ſuum B. Et ubi crurum AC, BC concurſus C ſucceſſive applicatur ad puncta quatuor reliqua D, E, F, G incidat concurſus crurum reliquorum AB & BA in puncta quatuor P, Q, R, S. Per puncta illa quatuor & quintum A deſcribatur ſectio Conica, & anguli præfati CAB, CBA ita rotentur ut crurum AB, BA concurſus percurrat ſectionem illam Conicam, & concurſus reliquorum crurum AC, BC deſcribet Curvam propoſitam per *Theorema* ſecundum.

Si vice puncti C datur poſitione recta BC quæ Curvam deſcribendam tangit in B, lineæ AD, AP coincident, & vice anguli DAP habebitur linea recta circa polum A rotanda.

Si Punctum duplex A infinite diſtat debebit Recta ad plagam puncti illius perpetuo dirigi & motu parallelo ferri interea dum angulus ABC circa polum B rotatur.

Deſcribi etiam poſſunt hæ Curvæ paulo aliter per *Theorema* tertium, ſed deſcriptionem ſimpliciorem poſuiſſe ſufficit.

Eadem methodo Curvas tertii, quarti & ſuperiorum Generum deſcribere licet, non omnes quidem ſed quotquot ratione aliqua commoda per motum localem deſcribi poſſunt. Nam Curvam aliquam ſecundi vel ſuperioris generis Punctum duplex non habentem commode deſcribere Problema eſt inter difficiliora numerandum.

VII. *Con-*

VII. *Conſtructio æquationum per deſcriptionem Curvarum.*

Curvarum uſus in Geometria eſt ut per earum interſectiones Problemata ſolvantur. Proponatur æquatio conſtruenda dimenſionum novem

$$x^9 * + bx^7 + cx^6 + dx^5 + ex^4 \underset{+m}{+ fx^3} + gx^2 + hx + k = 0.$$

Ubi *b*, *c*, *d*, &c. ſignificant quantitates quaſvis datas ſignis ſuis + & — affectas. Aſſumatur æquatio ad Parabolam cubicam $x^3 = y$, & æquatio prior, ſcribendo y pro x^3, evadet

$$y^3 + bxy^2 + cy^2 + dx^2y + exy + my + fx^3 + gx^2 + hx + k = 0,$$

æquatio ad Curvam aliam ſecundi Generis. Ubi *m* vel *f* deeſſe poteſt vel pro lubitu aſſumi. Et per harum Curvarum deſcriptiones & interſectiones dabuntur radices æquationis conſtruendæ. Parabolam cubicam ſemel deſcribere ſufficit.

Si æquatio conſtruenda per defectum duorum terminorum ultimorum hx & k reducatur ad ſeptem dimenſiones, Curva altera delendo *m*, habebit Punctum Duplex in principio Abſciſſæ, & inde facile deſcribi poteſt ut ſupra.

Si æquatio conſtruenda per defectum terminorum trium ultimorum $gx^2 + hx + k$ reducatur ad ſex dimenſiones, Curva altera delendo *f* evadet ſectio Conica.

Et ſi per defectum ſex ultimorum terminorum æquatio conſtruenda reducatur ad tres dimenſiones, incidetur in conſtructionem *Walliſianiam* per Parabolam cubicam & Lineam rectam.

Conſtrui etiam poſſunt æquationes per Hyperboliſmum Parabolæ cum diametro. Ut ſi conſtruenda ſit hæc æquatio dimenſionum novem termino penultimo carens,

$$a + cx^2 + dx^3 + ex^4 + fx^5 \underset{+m}{+ gx^6} + hx^7 + kx^8 + lx^9 = 0;$$

Aſſumatur æquatio ad Hyperboliſmum illum $x^2y = 1$, & ſcribendo y pro $\frac{1}{xx}$, æquatio conſtruenda vertetur in hanc

$$ay^3 + cy^2 + dxy^2 + ey + fxy + mx^2y + g + hx + kx^2 + lx^3 = 0,$$

quæ curvam ſecundi Generis deſignat cujus deſcriptione Problema ſolvetur. Et quantitatum *m* ac *g* alterutra hic deeſſe poteſt, vel pro lubitu aſſumi.

Per Parabolam cubicam & Curvas tertii Generis conſtruuntur etiam æquationes omnes dimenſionum non pluſquam duodecim, & per eandem Parabolam & Curvas quarti Generis conſtruuntur omnes dimenſionum non pluſquam quindecim; Et ſic deinceps in infinitum. Et Curvæ illæ tertii, quarti & ſuperiorum Generum deſcribi ſemper poſſunt inveniendo eorum puncta per Geometriam planam. Ut ſi conſtruenda ſit æquatio

$$x^{12} \ast + ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + kx + l = 0,$$

& deſcripta habeatur Parabola Cubica; ſit æquatio ad Parabolam illam Cubicam $x^3 = y$, & ſcribendo y pro x^3, æquatio conſtruenda vertetur in hanc

$$\begin{array}{lllll} y^4 & + axy^3 & + cx^2y^2 & + fx^2y & + ix^2 = 0, \\ & + b & + dx & + gx & + kx \\ & & + e & + h & + l \end{array}$$

quæ eſt æquatio ad Curvam tertii Generis cujus deſcriptione Problema ſolvetur. Deſcribi autem poteſt hæc Curva inveniendo ejus puncta per Geometriam planam, propterea quod indeterminata quantitas x non niſi ad duas dimenſiones aſcendit.

METHODUS DIFFERENTIALIS.

PROP. I.

SI *figuræ curvilineæ Abſciſſa componatur ex quantitate quavis data* A, *&* *quantitate indeterminata* x, *& Ordinata conſtet ex datis quotcunque quantitatibus* b, c, d, e, *&c. in totidem terminos hujus progreſſionis Geometricæ* x, x^2, x^3, x^4, *&c. reſpective ductis, & ad Abſciſſæ puncta totidem data erigantur Ordinatim applicatæ : dico quod Ordinatarum differentiæ primæ dividi poſſint per earum intervalla, & differentiarum ſic divi-*

ſarum differentiæ dividi poſſint per Ordinatarum binarum intervalla, & harum differentiarum ſic diviſarum differentiæ dividi poſſint per Ordinatarum ternarum intervalla, & ſic deinceps in infinitum.

Etenim ſi pro Abſciſſæ parte indeterminata x ponantur quantitates quævis datæ p, q, r, s, t, &c. ſucceſſive, & ad Abſciſſarum ſic datarum terminos erigantur Ordinatæ α, β, γ, δ, ϵ, &c. Hæ Abſciſſæ & Ordinatæ & Ordinatarum differentiæ diviſæ per Abſciſſarum differentias (quæ utique ſunt Ordinatarum intervalla) & quotorum differentiæ diviſæ per Ordinatarum alternarum differentias, & ſic deinceps, exhibentur per Tabulam ſequentem.

Abſciſſæ	Ordinatæ
$A+p$	$A+bp+cp^2+dp^3+ep^4=\alpha$
$A+q$	$A+bq+cq^2+dq^3+eq^4=\beta$
$A+r$	$A+br+cr^2+dr^3+er^4=\gamma$
$A+s$	$A+bs+cs^2+ds^3+es^4=\delta$
$A+t$	$A+bt+ct^2+dt^3+et^4=\epsilon$
Diviſor. Diff. Ord.	Quoti per diviſionem prodeuntes.
$p-q)\ \alpha-\beta$	$b+c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+p^2q+pq^2+q^3}=\zeta$
$q-r)\ \beta-\gamma$	$b+c\times\overline{q+r}+d\times\overline{qq+qr+rr}+e\times\overline{q^3+q^2r+qr^2+r^3}=\eta$
$r-s)\ \gamma-\delta$	$b+c\times\overline{r+s}+d\times\overline{rr+rs+ss}+e\times\overline{r^3+r^2s+rs^2+s^3}=\theta$
$s-t)\ \delta-\epsilon$	$b+c\times\overline{s+t}+d\times\overline{ss+st+tt}+e\times\overline{s^3+s^2t+st^2+t^3}=\kappa$
$p-r)\ \zeta-\eta$	$c+d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}=\lambda$
$q-s)\ \eta-\theta$	$c+d\times\overline{q+r+s}+e\times\overline{qq+qr+rr+qs+rs+ss}=\mu$
$r-t)\ \theta-\kappa$	$c+d\times\overline{r+s+t}+e\times\overline{rr+rs+ss+rt+st+tt}=\nu$
$p-s)\ \lambda-\mu$	$d+e\times\overline{p+q+r+s}=\xi.$
$q-t)\ \mu-\nu$	$d+e\times\overline{q+r+s+t}=\pi.$
$p-t)\ \xi-\pi$	$e=\sigma.$

PROP.

PROP. II.

Iiſdem poſitis, & quod numerus terminorum b, c, d, e, *&c. ſit finitus, dico quod Quotorum ultimus æqualis erit ultimo terminorum* b, c, d, e, *&c. et quod per Quotos reliquos dabuntur termini reliqui* b, c, d, e, *&c. et his datis dabitur Linea Curva generis Parabolici quæ per Ordinatarum omnium terminos tranſibit.*

Etenim in Tabula ſuperiore Quotus ultimus σ æqualis erat termino ultimo e. Et hic terminus ductus in ſummam datam $\overline{p+q+r+s}$, & ablatus de Quoto ξ relinquit terminum penultimum d. Et quantitates jam datæ $d\times\overline{p+q+r}+e\times\overline{pp+pq+qq+pr+qr+rr}$, ſi auferantur de Quoto λ, relinquunt terminorum antepenultimum c. Et quantitates jam datæ $c\times\overline{p+q}+d\times\overline{pp+pq+qq}+e\times\overline{p^3+ppq+pqq+q^3}$, ſi auferantur de Quoto ζ, relinquunt terminum b. Et ſimili computo ſi plures eſſent termini, colligerentur omnes per Quotorum Ordines totidem. Deinde quantitates datæ $bp+cpp+dp^3+ep^4$, ſi ſubducantur de Ordinata prima α, relinquunt Abſciſſæ terminum primum A. Et quantitas $A+bx+cx^2+dx^3+ex^4+$ &c. eſt Ordinata Curvæ generis Parabolici quæ per Ordinatarum omnium datarum terminos tranſibit, exiſtente Abſciſſa $A+x$.

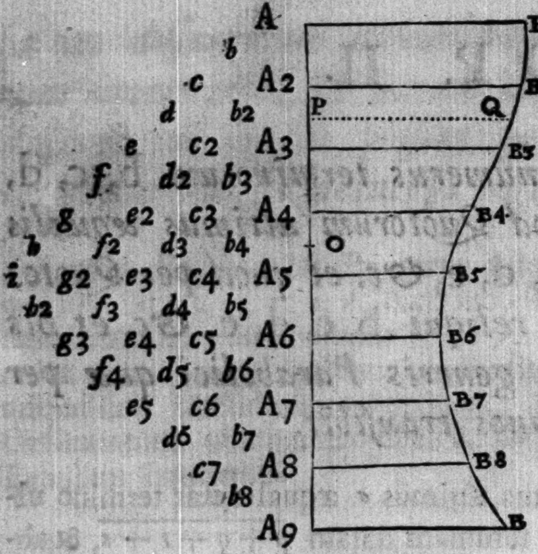
Ex his Propoſitionibus quæ ſequuntur facile colligi poſſunt.

PROP. III.

Si Recta aliqua AA9 *in æquales quotcunque partes* AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Erectarum AB, A2B2, A3B3, &c. quære differentias Primas, b, $b2$, $b3$, &c. Secundas c, $c2$, $c3$, &c. Tertias d, $d2$, $d3$, &c. et ſic deinceps uſque dum veneris ad ultimam differentiam, quæ hic ſit i.

Tunc



Tunc incipiendo ab ultima differentia excerpe medias differentias in alternis Columnis vel Ordinibus differentiarum, & Arithmetica media inter duas medias reliquarum, Ordine pergendo uſque ad Seriem primorum terminorum AB, A2B2, A3B3, &c. ſint hæc *k*, *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, &c. quorum ultimus ſignificet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas differentias; antepenultimus mediam trium antepenultimarum differentiarum, & ſic deinceps uſque ad primum quod erit vel medius terminorum A, A2, A3, &c. vel Arithmeticus medius inter duos medios. Prius accidit ubi numerus terminorum A, A2, A3, &c. eſt impar; poſterius ubi par.

CAS. I.

In Caſu priori, ſit A5B5 iſte medius terminus, hoc eſt, $A5B5 = k$, $\frac{b4+b5}{2} = l$, $c4 = m$, $\frac{d3+d4}{2} = n$, $e3 = o$, $\frac{f2+f3}{2} = p$, $g2 = q$, $\frac{h+h2}{2} = r$, $i = s$. Et erecta Ordinatim applicata PQ, dic $A5P = x$; & duc terminos hujus Progreſſionis

$$1 \times \frac{x}{1} \times \frac{x}{2} \times \frac{x^2-1}{3x} \times \frac{x}{4} \times \frac{x^2-4}{5x} \times \frac{x}{6} \times \frac{x^2-9}{7x} \times \frac{x}{8} \times \frac{x^2-16}{9x} \times \frac{x}{10} \times \frac{x^2-25}{11x} \times \frac{x}{12} \times \frac{x^2-36}{13x} \text{ \&c.}$$

in ſe continuo; & orientur termini

$$1.\ x.\ \frac{x^2}{2}.\ \frac{x^3-x}{6}.\ \frac{x^4-x^2}{24}.\ \frac{x^5-5x^3+4x}{120}.\ \frac{x^6-5x^4+4x^2}{720}.\ \frac{x^7-14x^5+49x^3-36x}{5040}.\ \text{\&c.}$$

per quos ſi termini ſeriei *k*, *l*, *m*, *n*, *o*, *p*, &c. reſpective multiplicentur, aggregatum factorum $k + xl + \frac{x^2}{2}m + \frac{x^3-x}{6}n + \frac{x^4-x^2}{24}o + \frac{x^5-5x^3+4x}{120}p + \text{\&c.}$ erit longitudo Ordinatim applicatæ PQ.

CAS. II.

In Caſu poſteriori, ſint A4B4, A5B5 duo medii termini, hoc eſt, ſit $\frac{A4B4+A5B5}{2} = k$, $b4 = l$, $\frac{c3+c4}{2} = m$, $d3 = n$, $e2 + e3 = o$, $f2 = p$, $\frac{g+g2}{2} = q$, & $h = r$.

& $b = r$. Et erecta Ordinatim applicata PQ, biſeca A4A5 in O, & dicto OP $= x$, duc Terminos hujus Progreſſionis $1 \times \frac{x}{1} \times \frac{xx - \frac{1}{4}}{2x} \times \frac{x}{3} \times \frac{xx - \frac{9}{4}}{4x} \times \frac{x}{5} \times \frac{xx - \frac{25}{4}}{6x} \times \frac{x}{7} \times \frac{xx - \frac{49}{4}}{8x}$, &c. in ſe continuo; et orientur termini $1.\ x.\ \frac{4xx-1}{8}.\ \frac{4x^3-x}{24}.\ \frac{16x^4-40x^2+9}{384}.$ &c. per quos ſi termini ſeries k, l, m, n, o, p, q, &c. reſpective multiplicentur, aggregatum factorum $k + xl + \frac{4x^2-1}{8}m + \frac{4x^3-x}{24}n + \frac{16x^4-40x^2+9}{384}o +$ &c. erit Longitudo Ordinatim applicatæ PQ.

Sed hic notandum eſt quod intervalla AA2, A2A3, A3A4, &c. hic ſupponantur eſſe unitates, & quod differentiæ colligi debent auferendo inferiores quantitates de ſuperioribus, A2B2 de AB, A3B3 de A2B2, $b2$ de b, &c. et faciendo ut ſint AB $-$ A2B2 $= b$, A2B2 $-$ A3B3 $= b2$, $b - b2 = c$, &c. adeoque quando differentiæ illæ hoc modo prodeunt negativæ ſigna earum mutanda ſunt.

PROP. IV.

Si recta aliqua in partes quotcunque inæquales AA2, A2A3, A3A4, A4A5, *&c. dividatur, & ad puncta diviſionum erigantur parallelæ* AB, A2B2, A3B3, *&c. Invenire Curvam Geometricam generis Parabolici quæ per omnium erectarum terminos* B, B2, B3, *&c. tranſibit.*

Sunto puncta data B, B2, B3, B4, B5, B6, B7, &c. et ad Abſciſſam quamvis AA7 demitte Ordinatas perpendiculariter BA, B2A2, &c.

Et fac $\frac{AB - A2B2}{AA2} = b$, $\frac{A2B2 - A3B3}{A2A3} = b2$,

$\frac{A3B3 - A4B4}{A3A4} = b3$, $\frac{A4B4 - A5B5}{A4A5} = b4$,

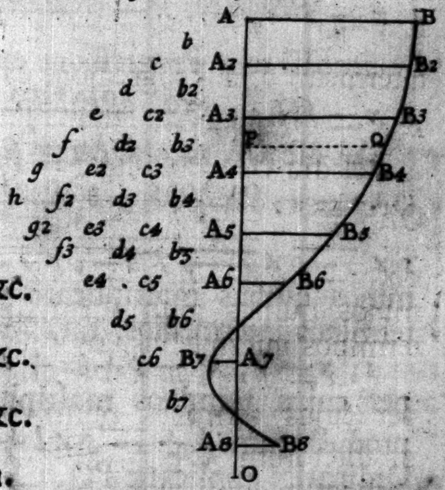
$\frac{A5B5 - A6B6}{A5A6} = b5$, $\frac{A6B6 - A7B7}{A6A7} = b6$,

$\frac{-A7B7 - A8B8}{A7A8} = b7$.

Deinde $\frac{b - b2}{AA3} = c$, $\frac{b2 - b3}{A2A4} = c2$, $\frac{b3 - b4}{A3A5} = c3$, &c.

Tunc $\frac{c - c2}{AA4} = d$, $\frac{c2 - c3}{A2A5} = d2$, $\frac{c3 - c4}{A3A6} = d3$, &c.

Et $\frac{d - d2}{AA5} = e$, $\frac{d2 - d3}{A2A6} = e2$, $\frac{d3 - d4}{A3A7} = e3$, &c.



Sic pergendum eſt ad ultimam differentiam.

Differentiis ſic collectis & diviſis per intervalla Ordinatim applicatarum; in alternis earum Columnis ſive Seriebus vel Ordinibus excerpe medias, incipiendo ab ultima, & in reliquis Columnis excerpe media Arithmetica inter duas medias, pergendo uſque ad ſeriem primorum terminorum, AB, A2B2, &c. Sunto hæc k, l, m, n, o, p, q, r, &c. quorum ultimus terminus ſignificet ultimam differentiam; penultimus medium Arithmeticum inter duas penultimas; antepenultimus mediam trium antepenultimarum, &c. Et primus k erit media Ordinatim applicata, ſi numerus datorum punctorum eſt impar; vel medium Arithmeticum inter duas medias, ſi numerus eorum eſt par.

CAS. I.

In Caſu priori, ſit A4B4 iſta media Ordinatim applicata, hoc eſt, ſit A4B4 $= k$, $\frac{b3+b4}{2} = l$, $c3 = m$, $\frac{d2+d3}{2} = n$, $e2 = o$, $\frac{f+f2}{2} = p$, $g = q$. Et erecta Ordinatim applicata PQ, & in Baſi AA5 ſumpto quovis puncto O, dic OP $= x$, & duc in ſe gradatim terminos hujus Progreſſionis

$$1 \times \overline{x - OA4} \times \overline{x - \frac{OA3+OA5}{2}} \times \frac{\overline{x-OA3} \times \overline{x-OA5}}{x - \frac{1}{2}\overline{OA3+OA5}} \times \overline{x - \frac{OA2+OA6}{2}} \times \text{\&c.}$$

et ortam Progreſſionem aſſerva; vel quod perinde eſt duc terminos hujus Progreſſionis

$$1 \times \overline{x-OA4} \times \overline{\overline{x-OA3} \times \overline{x-OA5}} \times \overline{\overline{x-OA2} \times \overline{x-OA6}} \times \overline{\overline{x-OA} \times \overline{x-OA7}} \times \text{\&c.}$$

in ſe gradatim, & terminos exinde ortos duc reſpective in terminos hujus Progreſſionis

$$1.\ x - \frac{+OA3+OA5}{2}.\ x - \frac{+OA2+OA6}{2}.\ x - \frac{+OA+OA7}{2},$$

&c. et orientur termini intermedii tota Progreſſione exiſtente

$$1.\ x - OA4.\ x^2 - \frac{+OA3+2OA4+OA5}{2}x + \frac{OA3+OA5}{2} \times OA4, \ \text{\&c.}$$

Vel dic OA $= \alpha$, OA2 $= \beta$, OA3 $= \gamma$, OA4 $= \delta$, OA5 $= \varepsilon$, OA6 $= \zeta$, OA7 $= \eta$: $\frac{OA3+OA5}{2} = \theta$, $\frac{OA2+OA6}{2} = \chi$, $\frac{OA+OA7}{2} = \lambda$. Et ex Progreſſione

$$1 \times \overline{x-\delta} \times \overline{\overline{x-\gamma} \times \overline{x-\varepsilon}} \times \overline{\overline{x-\beta} \times \overline{x-\zeta}} \times \overline{\overline{x-\alpha} \times \overline{x-\eta}}$$

&c. collige terminos quibus multiplicatis per $1.\ x-\theta,\ x-\chi,\ x-\lambda$, &c collige alios terminos intermedios, tota ſerie prodeunte

$$1,\ x-\delta,\ x^2 - \overline{\delta+\theta}\,x + \delta\theta,\ x^3 - \overline{\delta+2\theta}\,x^2 + \overline{\gamma\varepsilon+2\delta\theta}\,x - \gamma\delta\varepsilon, \ \text{\&c.}$$

per cujus terminos multiplica ſeries k, l, m, n, o, &c. Et aggregatum productorum $k + \overline{x-\delta} \times l + \overline{x^2 - \overline{\delta+\theta}\,x + \delta\theta} \times m +$ &c. erit longitudo Ordinatim applicatæ PQ.

CAS.

C A S. II.

In Caſu poſteriori, ſint $A4B4$, $A5B5$ duæ mediæ Ordinatim applicatæ; hoc eſt, $\frac{A4B4+A5B5}{2}=k$, $b4=l$, $\frac{c3+c4}{2}=m$, $d3=n$, $\frac{e2+e3}{2}=o$, $f2=p$, &c. Et alternorum k, m, o, q, &c. Coefficientes orientur ex multiplicatione terminorum hujus Progreſſionis in ſe

$$1\times\overline{x-OA4}\times\overline{x-OA5}\times\overline{x-OA3}\times\overline{x-OA6}\times\overline{x-OA2}\times\overline{x-OA7}\times\overline{x-OA}\times\overline{x-OA8}\ \&c.$$

Et reliquorum Coefficientes ex multiplicatione horum per terminos hujus Progreſſionis

$$x-\frac{+OA4+OA5}{2},\ x-\frac{+OA3+OA6}{2},\ x-\frac{+OA2+OA7}{2},\ x-\frac{+OA+OA8}{2},\ \&c.$$

Hoc eſt, erit $k+\overline{x-\frac{+OA4+OA5}{2}}\times l+\overline{x^2-\overline{OA4+OA5}\,x+OA4\times OA5}\times m$, &c. Ordinatim applicata PQ,

vel $PQ=k+\begin{matrix}x\\-\frac{1}{2}OA4\\-\frac{1}{2}OA5\end{matrix}\times l+\begin{matrix}x\\-OA4\end{matrix}\times+\begin{matrix}x\\-OA5\end{matrix}\times m+\begin{matrix}x\\-OA4\end{matrix}\times+\begin{matrix}x\\-OA5\end{matrix}\times+\begin{matrix}x\\-\frac{1}{2}OA3\\-\frac{1}{2}OA6\end{matrix}\times n$ &c.

Sive dic $x-\frac{+OA4+OA5}{2}=\pi$, $\overline{x-OA4}\times\overline{x-OA5}=\varrho$,

$$\varrho\times\overline{x-\frac{+OA3+OA6}{2}}=\sigma,\ \varrho\times\overline{x-OA3}\times\overline{x-OA6}=\tau,$$

$$\tau\times\overline{x-\frac{+OA2+OA7}{2}}=\upsilon,\ \tau\times\overline{x-OA2}\times\overline{x-OA7}=\varphi,$$

$$\varphi\times\overline{x-\frac{+OA+OA8}{2}}=\chi,\ \varphi\times\overline{x-OA}\times\overline{x-OA8}=\psi,$$

Et erit $k+\pi l+\varrho m+\sigma n+\tau o+\upsilon p+\varphi q+\chi r+\psi s=PQ$.

P R O P. V.

Datis aliquot terminis ſeriei cujuſcunque ad data intervalla diſpoſitis, invenire terminum quemvis intermedium quamproxime.

Ad rectam poſitione datam erigantur termini dati in dato angulo, interpoſitis datis intervallis, & per eorum puncta extima, per Propoſitiones præcedentes, ducatur linea Curva generis Parabolici. Hæc enim continget terminos omnes intermedios per ſeriem totam.

PROP. VI.

Figuram quamcunque Curvilineam quadrare quamproxime, cujus Ordinatæ aliquot inveniri poſſunt.

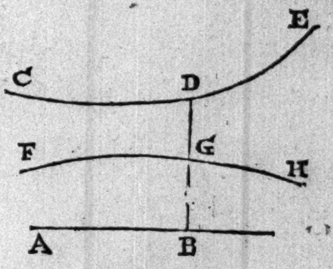
Per terminos Ordinatarum ducatur linea Curva generis Parabolici ope Propoſitionum præcedentium. Hæc enim figuram terminabit quæ ſemper quadrari poteſt, et cujus Area æquabitur Areæ figuræ propoſitæ quamproxime.

SCHOLIUM.

Utiles ſunt hæ Propoſitiones ad Tabulas conſtruendas per interpolationem Serierum, ut & ad ſolutiones Problematum quæ a quadraturis Curvarum dependent, præſertim ſi Ordinatarum intervalla & parva ſint & æqualia inter ſe, & Regulæ computentur, & in uſum reſerventur pro dato quocunque numero Ordinatarum. Ut ſi quatuor ſint Ordinatæ ad æqualia intervalla ſitæ, ſit A ſumma primæ & quartæ, B ſumma ſecundæ & tertiæ, & R intervallum inter primam & quartam, & Ordinata nova in medio omnium erit $\frac{9B-A}{16}$, & Area tota inter primam & quartam erit $\frac{A+3B}{8}R$.

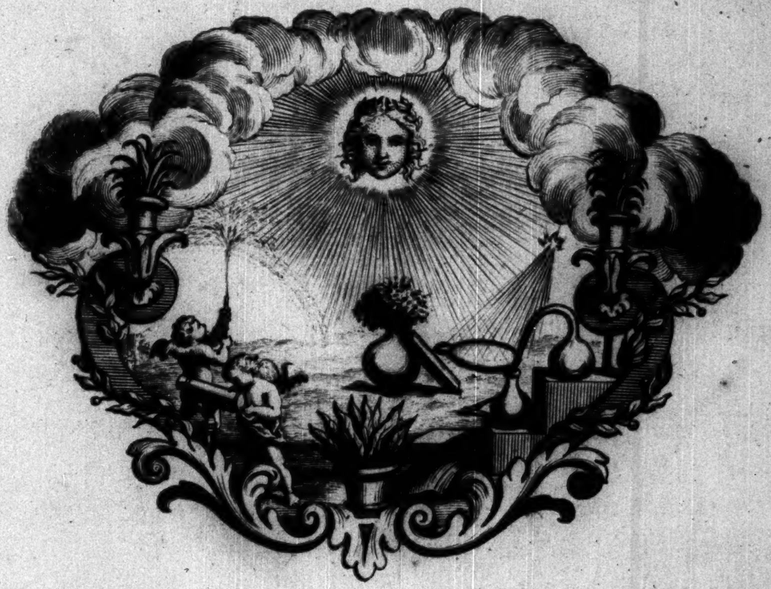
Et nota quod ubi Ordinatæ ſtant ad æquales ab invicem diſtantias, ſumendo ſummas Ordinatarum quæ ab Ordinata media hinc inde æqualiter diſtant, & duplum Ordinatæ mediæ, componitur Curva nova cujus Area per pauciores Ordinatas determinatur, & æqualis eſt Areæ Curvæ prioris quam invenire oportuit. Quinetiam ſi pro Ordinatis novis ſumantur ſumma Ordinatæ primæ & ſecundæ, et ſumma tertiæ & quartæ, et ſumma quintæ & ſextæ, & ſic deinceps; vel ſi ſumantur ſumma trium primarum Ordinatarum, & ſumma trium proximarum, & ſumma trium quæ ſunt deinceps; vel ſi ſumantur ſummæ quaternarum Ordinatarum, vel ſummæ quinarum: Area Curvæ novæ æqualis erit Areæ Curvæ primo propoſitæ. Et ſic habitis Curvæ quadrandæ Ordinatis quotcunque quadratura ejus ad quadraturam Curvæ alterius per pauciores Ordinatas reducetur.

Per data vero puncta quotcunque non ſolum Curvæ lineæ generis *Parabolici*, ſed etiam Curvæ aliæ innumeræ diverſorum generum duci poſſunt.



Sunto CDE, FGH Curvæ duæ Abſciſſam habentes communem AB, et Ordinatas in eadem recta jacentes BD, BG; & relatio inter has Ordinatas definiatur per æquationem quamcunque. Dentur puncta quotcunque per quæ Curva CDE tranſire debet, & per æquationem illam dabuntur puncta totidem nova per quæ Curva FGH tranſibit. Per Propoſitiones ſuperiores deſcribatur Curva FGH generis *Parabolici* quæ per puncta illa omnia nova tranſeat, & per æquationem eandem dabitur Curva CDE quæ per puncta omnia primo data tranſibit.

FINIS.



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